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SOME STRUCTURAL CONSIDERATIONS ON THE THEORY OF GRAVITATIONAL FIELD IN FINSLER SPACES

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Some structural generalizations of the Finslerian gravitational field are considered by generalizing the internal variables in various ways.

1. INTRODUCTION

From the vector bundle-like standpoint (Miron and Anastasiei[4]), the Finslerian gravitational field is regarded as a unified field of the base (x)-field spanned by points $\{x\}$ ($x = x^\alpha$; $\alpha = 1, 2, 3, 4$) and the internal (y)-field spanned by vectors $\{y\}$ ($y = y^i$; $i = 1, 2, 3, 4$), where the (x)-field is the base manifold and the (y)-field is the fibre at each point. Of course, the (x)-field is nothing else than the Einstein's gravitational field itself.

The Finslerian features are characterized by the transformation law of y, i.e., $\bar{y}^i = A^i_j(x)\,y^j$, where A^i_j is not chosen as $\dfrac{\partial \bar{x}^i}{\partial x^j}$, in general. Moreover, this transformation can be adjusted to the so-called gauge transformation $\bar{y}^i = Y^i(x, y)$, $\bar{x}^\alpha = X^\alpha(x^\beta)$, from which we can obtain

$$\left.\begin{aligned} &\frac{\partial}{\partial x^\alpha} = X^\beta_\alpha \frac{\partial}{\partial \bar{x}^\beta} + \frac{\partial Y^i}{\partial x^\alpha}\frac{\partial}{\partial \bar{y}^i}, \quad \left(X^\beta_\alpha = \frac{\partial X^\beta}{\partial x^\alpha}\right), \\ &\frac{\partial}{\partial y^i} = Y^j_i \frac{\partial}{\partial \bar{y}^j}, \quad \left(Y^j_i = \frac{\partial Y^j}{\partial y^i}\right). \end{aligned}\right\} \qquad \text{... (1)}$$

If the gauge transformation is newly rewritten as $\bar{y}^i = Y^i_j(x, y)\,y^j$ with

$$Y^i_j(x, y) = Y^i_j(0, 0) + \frac{\partial Y^i_j}{\partial x^\alpha}dx^\alpha + \frac{\partial Y^i_j}{\partial y^k}dy^k, \qquad \text{... (2)}$$

then the new formulation of the intrinsic parallelism of y can be obtained as follows :

$$\bar{\delta} y^i = dy^i + K^i_{j\alpha} y^j dx^\alpha + K^i_{jk} y^j dy^k \,(= 0), \qquad \text{... (3)}$$

where $\quad dy^i = \bar{y}^i - Y^i_j(0, 0)\, y^j$, $K^i_{j\alpha} = -\dfrac{\partial Y^i_j}{\partial x^\alpha}$ and $K^i_{jk} = -\dfrac{\partial Y^i_j}{\partial y^k}$.

Those connection coefficients represent the concept of unified gauge fields and are assumed to obey the gauge transformations (1). This case has been fully considered by the author[2].

The quantity δy forms the adapted frame in the total space of the vector bundle, so that some structural generalizations can be considered by generalizing δy itself. From this standpoint, in this paper, we shall consider some generalizations of the Finslerian structure.

2. On the Finslerian Structure

As usual, the adapted frame is set as follows :

$$d\zeta^A = (dx^\alpha,\ \delta y^i = dy^i + N^i_\alpha dx^\alpha),$$

$$\frac{\partial}{\partial \zeta^A} = \left(\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N^i_\alpha \frac{\partial}{\partial y^i},\ \frac{\partial}{\partial y^i} \right), \qquad \text{... (4)}$$

where N^i_α means the nonlinear connection. In the case of (3), the adapted frame (4) is generalized in the form

$$(dx^\alpha,\ \bar{\delta} y^i = P^i_k dy^k + Q^i_\alpha dx^\alpha),$$

$$\left(\frac{\bar{\delta}}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - \bar{N}^i_\alpha \frac{\partial}{\partial y^i},\ \frac{\bar{\delta}}{\delta y^i} = (P^{-1})^j_i \frac{\partial}{\partial y^j} \right), \qquad \text{... (5)}$$

where $P^i_k = \delta^i_k + K^i_{jk} y^j$, $Q^i_\alpha = K^i_{j\alpha} y^j$, $\bar{N}^i_\alpha = (P^{-1})^i_j Q^j_\alpha$. The connection structure is introduced by

$$\frac{\nabla_\partial}{\partial \zeta^C} \frac{\partial}{\partial \zeta^B} = \Gamma^A_{BC} \frac{\partial}{\partial \zeta^A};\ \Gamma^A_{BC} = (F^\alpha_{\beta\gamma},\ F^i_{j\gamma},\ C^\alpha_{\beta k},\ C^i_{jk}), \qquad \text{... (6)}$$

where

$$\frac{\nabla_\delta}{\delta x^\gamma} \frac{\delta}{\delta x^\beta} = F^\alpha_{\beta\gamma} \frac{\delta}{\delta x^\alpha},\ \frac{\nabla_\partial}{\partial y^k} \frac{\partial}{\partial y^j} = C^i_{jk} \frac{\partial}{\partial y^i},\ \text{etc.} \qquad \text{... (7)}$$

Correspondingly, from (5), we can introduce another generalized connection structure in the form $\dfrac{\nabla_{\bar{\delta}}}{\delta x^\gamma} \dfrac{\bar{\delta}}{\delta x^\beta} = \bar{F}^\alpha_{\beta\gamma} \dfrac{\bar{\delta}}{\delta x^\alpha}$, $\dfrac{\nabla_{\bar{\delta}}}{\delta y^k} \dfrac{\bar{\delta}}{\delta y^j} = \bar{C}^i_{jk} \dfrac{\bar{\delta}}{\delta y^i}$, etc.

And the metrical structure is given by

$$G = G_{AB}\, d\zeta^A \otimes d\zeta^B = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + g_{ij}\, \delta y^i \otimes \delta y^j, \qquad \text{... (8)}$$

which can be generalized in the case of (5) as follows :

$$\overline{G} = \overline{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + \overline{g}_{ij}\, \overline{\delta}y^i \otimes \overline{\delta}y^j . \qquad \text{... (9)}$$

On the other hand, in the case of tangent bundle with $[x^i,\ y^i = \dfrac{dx^i}{dt}$ (velocity)], the equation $\delta y^i = 0$ gives the equation of motion, etc., so that the nonlinear connection N^i_j appearing in the equation $\dfrac{\delta y^i}{dt} = \dfrac{dy^i}{dt} + N^i_j \dfrac{dx^j}{dt} = 0$ can be determined from the Lagrangian $L(x, y)$ $(\delta \int L(x, y)\, dt = 0)$ in the form

$$N^i_j = \frac{\partial G^i}{\partial y^j}\ ;\ G^i = \frac{1}{4} g^{ij} \left(\frac{\partial^2 L}{\partial y^j\, \partial x^k} y^k - \frac{\partial L}{\partial x^j} \right). \qquad \text{... (10)}$$

In the case of the conformally Riemannian metric $g_{ij} = e^{2\sigma(x,y)} \gamma_{ij}(x)$, the nonlinear connection N^i_j is often put in the form $N^i_j = \left\{ {i \atop jk} \right\} y^k$, where $\left\{ {i \atop jk} \right\}$ is the Christoffel symbol formed with $\gamma_{ij}(x)$ (Watanabe *et al.*[5]).

3. On the Generalized Structure I

If many kinds of vectors $y^{(\alpha)}$ (α = 1, 2, ..., n) are adopted as the independent internal variables and their interactions are taken into account, then the adapted frame $\tilde{\delta}\, y^{(\alpha)}$ can be given by, by the analogy with the base connection in the theory of higher order spaces (Kawaguchi[3]),

$$\tilde{\delta}\, y^{(\alpha)i} = dy^{(\alpha)i} + \sum_{\beta=1}^{\alpha-1} R^{(\alpha)i}_{(\beta)k}\, dy^{(\beta)k} + M^{(\alpha)i}_j\, dy^{(\alpha)j} + R^{(\alpha)i}_{(0)\alpha}\, dx^\alpha$$

$$= M^{(\alpha)i}_j \left[dy^{(\alpha)j} + \sum_{\beta=1}^{\alpha-1} \Lambda^{(\alpha)j}_{(\beta)k}\, dy^{(\beta)k} + \Lambda^{(\alpha)j}_{(0)\alpha}\, dx^\alpha \right]. \qquad \text{... (11)}$$

In the case of $\alpha = 1$,

$$\tilde{\delta}\, y^{(1)i} = dy^{(1)i} + M^{(1)i}_j\, dy^{(1)j} + R^{(1)i}_{(0)\alpha}\, dx^\alpha$$

$$= M^{(1)i}_j \left[dy^{(1)j} + \Lambda^{(1)j}_{(0)\alpha}\, dx^\alpha \right], \qquad \text{... (12)}$$

which can be compared with (3) and (5) by putting $P^i_k = M^{(1)i}_k$ and $Q^i_\alpha = M^{(1)i}_j \Lambda^{(1)j}_{(0)\alpha}$. From (11), we can obtain

$$M^{(\alpha)i}_j\, dy^{(\alpha)j} = \tilde{\delta}\, y^{(\alpha)i} - \sum_{\beta=1}^{\alpha-1} \Pi^{(\alpha)i}_{(\beta)k}\, dy^{(\beta)k} - \Pi^{(\alpha)i}_{(0)\alpha}\, dx^\alpha, \qquad \text{... (13)}$$

where $\Pi^{(\alpha)i}_{(\beta)j} = M^{(\alpha)i}_k \Lambda^{(\alpha)k}_{(\beta)j}$, $\Pi^{(\alpha)i}_{(0)\alpha} = M^{(\alpha)i}_k \Lambda^{(\alpha)k}_{(0)\alpha}$. In general, the following conditions hold good ($\alpha, \beta, \gamma = 0, 1, 2, \ldots, n$) :

$$M_j^{(\alpha)i} \Lambda_{(\gamma)k}^{(\alpha)j} + \Pi_{(\gamma)j}^{(\alpha)i} M_k^{(\gamma)j} + \sum_{\beta=\gamma+1}^{\alpha-1} \Pi_{(\beta)j}^{(\alpha)i} M_l^{(\beta)j} \Lambda_{(\gamma)k}^{(\beta)l} = 0. \qquad \ldots (14)$$

In this case, the whole adapted frame is given by

$$(dx^\alpha, \tilde{\delta} y^{(\alpha)i} = M_j^{(\alpha)i} \left(\Lambda_{(0)\alpha}^{(\alpha)j} dx^\alpha + \sum_{\beta=1}^{\alpha-1} \tilde{\Lambda}_{(\beta)k}^{(\alpha)j} dy^{(\beta)k} \right)$$

$$= N_\alpha^{(\alpha)i} dx^\alpha + \sum_{\beta=1}^{\alpha-1} \overline{\Psi}_{(\beta)k}^{(\alpha)i} dy^{(\beta)k}), \qquad \ldots (15)$$

$$\left(\frac{\tilde{\delta}}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - \sum_{\beta=\alpha+1}^{n} (\overline{\Psi}^{-1})_{(\alpha)i}^{(\beta)k} N_\alpha^{(\alpha)i} \frac{\partial}{\partial y^{(\beta)k}}, \ \frac{\tilde{\partial}}{\partial y^{(\alpha)k}} = \sum_{\beta=\alpha+1}^{n} (\overline{\Psi}^{-1})_{(\alpha)k}^{(\beta)l} \frac{\partial}{\partial y^{(\beta)l}} \right),$$

where $N_\alpha^{(\alpha)i} = M_j^{(\alpha)i} \Lambda_{(0)k}^{(\alpha)j}$, $\Psi_{(\beta)k}^{(\alpha)i} = M_j^{(\alpha)i} \tilde{\Lambda}_{(\beta)k}^{(\alpha)j}$ and $\tilde{\Lambda}_{(\beta)k}^{(\alpha)j} = \delta_{(\beta)}^{(\alpha)} \delta_k^j + \Lambda_{(\beta)k}^{(\alpha)j}$.

Then, the connection structure is given by

$$\tilde{\Gamma}_{BC}^{A} = \left(\tilde{F}_{\beta\gamma}^{\alpha}, \ \tilde{F}_{(\beta)j\gamma}^{(\alpha)i}, \ \tilde{C}_{\beta\,(\alpha)i}^{\alpha}, \ \tilde{C}_{(\beta)j\,(\gamma)k}^{(\alpha)i} \right), \qquad \ldots (16)$$

and the metrical structure is written as

$$\tilde{G} = \tilde{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + \tilde{g}_{(\alpha)i(\beta)j}\, \tilde{\delta} y^{(\alpha)i} \otimes \tilde{\delta} y^{(\beta)j}. \qquad \ldots (17)$$

These most generalized formulations can be specialized in various ways : For example, we can reduce $\tilde{F}_{(\beta)j\gamma}^{(\alpha)i}$ and $\tilde{C}_{(\beta)j(\gamma)k}^{(\alpha)i}$ to $\tilde{F}_{(\alpha)j\gamma}^{(\alpha)i}$ and $\tilde{C}_{(\alpha)j(\gamma)k}^{(\alpha)i}$ and further $\tilde{C}_{(\alpha)j(\gamma)k}^{(\alpha)i}$ to $\tilde{C}_{(\alpha)j(\alpha)k}^{(\alpha)i}$, etc.

4. On the Generalized Structure II

If we adopt one scalar parameter λ and construct the second order vector bundle-like structure, then we can put the adapted frame in the form, different from the time-dependent Lagrange geometry (Anastasiei and Kawaguchi[1]),

$$(dx^\alpha, \ \delta y^i = dy^i + N_\alpha^i dx^\alpha, \ \delta\lambda = d\lambda + M_\alpha^0 dx^\alpha + L_i^0 dy^i),$$

$$\left(\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N_\alpha^i \frac{\partial}{\partial y^i} - M_\alpha^0 \frac{\partial}{\partial \lambda}, \ \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - L_i^0 \frac{\partial}{\partial \lambda}, \ \frac{\partial}{\partial \lambda} \right). \qquad \ldots (18)$$

Three kinds of nonlinear connections N_α^i, M_α^0 and L_i^0 must be introduced. Then, in general, nine connection coefficients are introduced :

$$\Gamma_{BC}^{A} = (F_{\beta\gamma}^{\alpha}, \ F_{j\gamma}^{i}, \ F_{0\gamma}^{0}, \ C_{\beta k}^{\alpha}, \ C_{jk}^{i}, \ C_{0k}^{0}, \ B_{\beta 0}^{\alpha}, \ B_{j0}^{i}, \ B_{00}^{0}). \qquad \ldots (19)$$

If we focus our attention on the λ-dependent effects, then we should take out those terms such as the nonlinear connections M^0_α, L^0_i and the connection coefficients $B^\alpha_{\beta 0}$, B^i_{j0} and those torsion and curvature tensors constructed from them. Since the metrical structure in this case is given by

$$G = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + g_{ij}\, \delta y^i \otimes \delta y^j + g_{00}\, \delta\lambda \otimes \delta\lambda, \qquad \dots (20)$$

if we put $g_{00} =$ const., then we can put $F^0_{0\gamma} = C^0_{0k} = B^0_{00} = 0$ in some special cases. By doing so, we could consider the λ-dependent effects conveniently.

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