Indian J. pure appl. Math., 27(7) : 659-666, July 1996

ALMOST SURE CONVERGENCE OF SET-VALUED GENERALIZED MARTINGALES AND SUBMARTINGALES

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(*Received* 12 *September* 1995; *accepted* 22 *February* 1996)

Dam[1] proved Doob's theorem for set-valued (sub-) martingales under analogous conditions for the Mosco-convergence (see Mosco[10], Hiai[6]). We prove a generalization of this theorem for so called set-valued generalized (sub-) martingales. The generalization comes from a modified (sub-) martingale relation by a limitation matrix. Our conditions on the set-valued processes are the same as in the paper of Dam.

1. INTRODUCTION

The aim of this paper is a generalization of a recently publicated paper of Dam[1]. The almost sure convergence of a class of set-valued martingales in the Hausdorff-metric has been discussed by Hiai and Umegaki[7] and Hiai[5]. Later on, the authors Dam and Tien[2] have treated the almost sure convergence of a larger class than that of set-valued martingales, the so called set-valued amarts. In this paper we present the Mosco-type convergence (see Mosco[10], Hiai[6]) for generalized martingales and submartingales taking values in $\mathcal{P}_c(\mathcal{X})$, where $\mathcal{X}$ is a reflexive separable Banach-space and $\mathcal{P}_c(\mathcal{X})$ is the class of all convex, closed, bounded non-empty subsets of $\mathcal{X}$. The generalization comes from a modified (sub-) martingale relation with a limitation matrix. We prove the almost sure convergence of set-valued generalized martingales and submartingales under the same conditions on the set-valued stochastic process as Dam. We shall show that under some conditions on the modified (sub-) martingale relation every set-valued martingale (submartingale), which is L^1-bounded, converges almost surely in Mosco-sense. The conditions on the given limitation matrix are always fulfilled in the situation of a set-valued martingale (submartingale).

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{X}, \|\cdot\|)$ a real separable and reflexive Banach-space with dual space $\mathcal{X}^*$. For each $X \subset \mathcal{X}$, $cl\ X$, $\overline{co}X$ will denote the norm-closure and the closed, convex hull of X, respectively. Let $\mathcal{P}(\mathcal{X})$ (resp. $\mathcal{P}_c(\mathcal{X})$) denote the family of all non-empty, closed, bounded (resp. non-empty, closed, convex, bounded) subsets of $\mathcal{X}$.

The convergence in the Mosco-sense is the following (see Mosco[10]). Let $\{X_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}_c(\mathcal{X})$. Denote

$$s - \liminf_{n \to \infty} X_n = \left\{ x \in \mathcal{X} : \exists\, x_n \in X_n : \lim_{n \to \infty} \| x_n - x \| = 0 \right\},$$

$$w - \limsup_{n \to \infty} X_n = \left\{ x \in \mathcal{X} : \exists\, x_k \in X_{nk} : x_k \xrightarrow{w} x \text{ (converges weakly)} \right\}.$$

We say that $\{X_n : n \in \mathbb{N}\}$ converges to X in the Mosco-sense and write $X_n \to X$ if

$$s - \liminf_{n \to \infty} X_n = w - \limsup_{n \to \infty} X_n = X.$$

For every $X \subset \mathcal{X}$, denote $\|X\| := \sup_{x \in X} \|x\|$. The set-valued function $F : \Omega \to \mathcal{P}(\mathcal{X})$ is called to be $\mathcal{A}$-measurable if for every open set O in $(\mathcal{X}, \|\cdot\|)$ we have

$$F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \phi\} \in \mathcal{A}.$$

We denote by $\mathcal{M}[\Omega, \mathcal{A}, P; \mathcal{X}] = \mathcal{M}[\Omega; \mathcal{X}]$ the family of all measurable set-valued functions $F : \Omega \to \mathcal{P}(\mathcal{X})$ and we set

$$\mathcal{L}^1[\Omega; \mathcal{X}] = \left\{ F \in \mathcal{M}[\Omega; \mathcal{X}] : \int_\Omega \| F(\omega) \| \, dP < \infty \right\},$$

$$\mathcal{L}^1_c[\Omega; \mathcal{X}] = \left\{ F \in \mathcal{L}^1[\Omega; \mathcal{X}] : F(\omega) \in \mathcal{P}_c(\mathcal{X}) \ a.s. \right\}.$$

We further denote by $\mathcal{L}^1(\Omega, \mathcal{A}, P; \mathcal{X}) = \mathcal{L}^1(\Omega; \mathcal{X})$ the Banach-space of all measurable functions $f : \Omega \to \mathcal{X}$ such that the norm

$$\|f\|_1 = \int_\Omega \| f(\omega) \| \, dP(\omega)$$

is finite.

For $F \in \mathcal{M}[\Omega; \mathcal{X}]$, let

$$S_F^1 = \{f \in L^1(\Omega, \mathcal{A}, P; \mathcal{X}) : f(\omega) \in F(\omega) \text{ a.s.}\}.$$

Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and besides S_F^1 defined on $(\Omega, \mathcal{A}, P)$, we take on $(\Omega, \mathcal{B}, P)$ the family

$$S_F^1(\mathcal{B}) = \{f \in L^1(\Omega, \mathcal{B}, P; \mathcal{X}) : f(\omega) \in F(\omega) \text{ a.s.}\}.$$

Recall that for $f \in L^1(\Omega; \mathcal{X})$ the conditional expectation of f relative to $\mathcal{B}$ is given as a function $E(f \mid \mathcal{B}) \in \mathcal{L}^1(\Omega, \mathcal{B}, P; \mathcal{X})$ such that

$$\int_B E(f \mid \mathcal{B})\, dP = \int_B f dP \text{ for all } B \in \mathcal{B}.$$

If $F \in \mathcal{M}[\Omega; \mathcal{X}]$ with $S_F^1 \neq \phi$, then by virtue of Theorem 5.1 of Hiai and Umegaki[7] there exists a unique (in the a.s. sense) $\mathcal{B}$-measurable function $E(F \mid \mathcal{B})$ satisfying

$$S^1_{E[F \mid \mathcal{B}]}(\mathcal{B}) = cl\,\{E(f \mid \mathcal{B}) : f \in S_F^1\},$$

where on the right-hand side we have taken the closure in the norm topology of $L^1(\Omega; \mathcal{X})$.

We call $E(F \mid \mathcal{B})$ the (set-valued) conditional expectation of F relative to $\mathcal{B}$.

3. Basic Definition and Results

At first we introduce generalized martingales and submartingales.

Definition 3.1 — Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be an increasing sequence of sub-σ-fields of $\mathcal{A}$ and let $\{F_n : n \in \mathbb{N}\}$ be a sequence of set-valued functions $F_n \in \mathcal{L}_1[\Omega; \mathcal{A}_n, P; \mathcal{X}]$. Further let $A = [a_{n,k}]_{k=1,\dots,n}^{n=1,\dots,\infty} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ be an infinite lower triangular real matrix. We say that $\{(F_n, \mathcal{A}_n) : n \in \mathbb{N}\}$ is a generalized set-valued (A-) martingale (resp. submartingale) if

$$\sum_{k=1}^{n} a_{n,k} F_k = E[F_{n+1} \mid \mathcal{A}_n] \text{ a.s.} \qquad \dots (3.1)$$

resp.
$$\sum_{k=1}^{n} a_{n,k} F_k \subset E[F_{n+1} \mid \mathcal{A}_n] \text{ a.s.} \qquad \dots (3.2)$$

for all $n \in \mathbb{N}$. The sum is defined as in Hiai and Umegaki[7], [p. 155, nos. (1.2) and (1.3)].

Remark 3.2 : A set-valued martingale (submartingale) is a genralized set-valued martingale (submartingale) with $A = [\delta_{n,k}]$ (Kronecker-symbol).

We need some basic tools from limitation theory for our main result.

Definition 3.3 — Let $A = [a_{n,k}]_{k=1,\dots,n}^{n=1,\dots,\infty} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be an infinite lower triangular matrix and $(\mathcal{X}, \|\cdot\|)$ a real Banach-space. We call A regular if for all convergent sequences $x = \{x_n : n \in \mathbb{N}\}$ in $(\mathcal{X}, \|\cdot\|)$ the sequence $\{\sigma_n^A(x) : n \in \mathbb{N}\} = \left\{\sum_{k=1}^{n} a_{n,k} x_k : n \in \mathbb{N}\right\}$ is also convergent in $(\mathcal{X}, \|\cdot\|)$ to the same limit.

This property can be characterized by the elements of the given matrix (see Zeller and Beekmann[14] or Peyerimhoff[13]).

Theorem 3.4 (Toeplitz) — Let $A = [a_{n,k}]$ be an infinite lower triangular real matrix. A is regular if and only if

$$\lim_{n\to\infty} a_{n,k} = 0 \text{ for all } k \in \mathbb{N}; \qquad \dots (3.3)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1 \qquad \dots (3.4)$$

and

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |a_{n,k}| < \infty \qquad \dots (3.5)$$

hold.

For our result we need the convergence equivalence property of A.

Definition 3.5 — Let $A = [a_{n,k}]$ be an infinite lower triangular real matrix and $(\mathcal{X}, \|\cdot\|)$ be a real Banach-space. A is called convergence equivalent if $\{x_n : n \in \mathbb{N}\}$ converges if and only if $\{\sigma_n^A(x) : n \in \mathbb{N}\}$ is convergent in $(\mathcal{X}, \|\cdot\|)$ and the limits are equal.

Remark 3.6 : A convergent equivalent matrix is regular.

Definition 3.7 — Let $A = [a_{n,k}]$ be an infinite lower triangular real matrix. We say A has the transition property (TP) if there exists a convergence equivalent matrix $B = [b_{n,k}]$ such that

$$b_{n,k} = b_{n+1,k} + b_{n+1,n+1}\, a_{n,k} \qquad \dots (3.6)$$

holds for all $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$.

Examples 3.8 — (1) $A = [\delta_{n,k}]$ (Kronecker-symbold) has TP.

(2) Every convergence equivalent Riesz-method (see Faulstich[3]) (R, p) has TP. In this case is $B = A$ suitable.

Finally we mention the convergence theorem for set-valued martingales and submartingales (see Dam[1]).

Theorem 3.9 — Let $(\mathcal{X}, \|\cdot\|)$ be a reflexive, separable real Banach-space and let $\{(F_n, \mathcal{A}_n) : n \in \mathbb{N}\}$, $F_n \in \mathcal{L}_c^1[\Omega, \mathcal{A}_n, P, \mathcal{X}]$ be a set-valued submartingale such that

$$\sup_{n \in \mathbb{N}} \int_{\Omega} \| F_n(w) \| \, dP(w) < \infty$$

holds. Then there exists a random element $F \in \mathcal{L}_c^1[\Omega, \mathcal{A}, P, \mathcal{X}]$ such that $F_n \to F$ a.s. where $\mathcal{A} := \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n \right)$.

4. The Main Result

Theorem 4.1 — Let $(\mathcal{X}, \|\cdot\|)$ be a reflexive, separable real Banach-space and let $\{(F_n, \mathcal{A}_n) : n \in \mathbb{N}\}$; $F_n \in \mathcal{L}_c^1[\Omega, \mathcal{A}_n, P; \mathcal{X}]$ be a process such that

$$\sup_{n \in \mathbb{N}} \int_{\Omega} \| F_n(w) \| \, dP(w) < \infty. \qquad \text{... (4.7)}$$

Further let $A = [a_{n,k}]$ be a matrix with (TP) such that

$$\sum_{k=1}^{n} a_{n,k} F_k \subset E[F_{n+1} \mid \mathcal{A}_n] \text{ a.s.} \quad (n \in \mathbb{N}). \qquad \text{... (4.8)}$$

Then there exists a random element $F \in \mathcal{L}_c^1[\Omega, \mathcal{A}, P; \mathcal{X}]$ such that $F_n \to F$ a.s. where $\mathcal{A} = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n \right)$.

Proof : Starting with the given process $\{F_n : n \in \mathbb{N}\}$ and the matrix A we construct a new process $\{G_n : n \in \mathbb{N}\}$ as follows

$$G_n := \sum_{k=1}^{n} b_{n,k} F_k \quad (n \in \mathbb{N}) \qquad \text{... (4.9)}$$

where B is convergence equivalent matrix that satisfies (3.6). Because of the regularity of B and $F_n \in \mathcal{L}_c^1[\Omega, \mathcal{A}_n, P; \mathcal{X}]$ we get

$$G_n \in \mathcal{L}_c^1[\Omega, \mathcal{A}_n, P, \mathcal{X}] \quad (n \in \mathbb{N}).$$

Further we can conclude

$$\sup_{n \in \mathbb{N}} \int_{\Omega} \| G_n(w) \| \, dP(w) = \sup_{n \in \mathbb{N}} \int_{\Omega} \left\| \sum_{k=1}^{n} b_{n,k} F_k(w) \right\| dP(w)$$

$$\leq \sup_{n \in \mathbb{N}} \int_{\Omega} \sum_{k=1}^{n} | b_{n,k} | \, \| F_k(w) \| \, dP(w)$$

$$= \sup_{n \in \mathbb{N}} \sum_{k=1}^{n} |b_{n,k}| \int_{\Omega} \| F_k(w) \| \, dP(w)$$

$$\leq \left(\sup_{m \in \mathbb{N}} \sum_{k=1}^{m} |b_{m,k}| \right) \cdot \sup_{n \in \mathbb{N}} \int_{\Omega} \| F_n(w) \| \, dP(w) < \infty$$

because of (3.5) and (3.7).

For the conditional expectation $E[G_{n+1} | \mathcal{A}_n]$ we can conclude (see Hiai and Umegaki[7], §5)

$$E[G_{n+1} | \mathcal{A}_n] = E\left[\sum_{k=1}^{n+1} b_{n+1,k} F_k | \mathcal{A}_n \right]$$

$$= \sum_{k=1}^{n} b_{n+1,k} E[F_k | \mathcal{A}_n] + b_{n+1,n+1} E[F_{n+1} | \mathcal{A}_n]$$

$$\supset \sum_{k=1}^{n} b_{n+1,k} F_k + b_{n+1,n+1} \sum_{k=1}^{n} a_{n,k} F_k$$

$$= \sum_{k=1}^{n} (b_{n+1,k} + b_{n+1,n+1} a_{n,k}) F_k$$

$$= \sum_{k=1}^{n} b_{n,k} F_k = G_n.$$

Therefore $\{(G_n, \mathcal{A}_n) : n \in \mathbb{N}\}$ is a set-valued submartingale that satisfies the condition of Theorem (3.9). It exists a random element $G \in \mathcal{L}_c^1[\Omega, \mathcal{A}, P; \mathcal{X}]$ such that $G_n \to G$ a.s. where $\mathcal{A} = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n \right)$.

We show now that

$$s - \liminf_{n \to \infty} F_n = w - \limsup_{n \to \infty} F_n = G$$

holds. Let $\varphi \in G$ be arbitrary chosen. For every $n \in \mathbb{N}$ there exists a $\varphi_n \in G_n$ such that $\lim_{n \to \infty} \| \varphi_n - \varphi \| = 0$. The relation $\varphi_n \in G_n$ implies $\varphi_n = \sum_{k=1}^{n} b_{n,k} \xi_k \, (\xi_k \in F_k)$ for all $n \in \mathbb{N}$. Because of the convergence equivalence of B we get $\lim_{n \to \infty} \| \varphi_n - \varphi \| = 0$ if and only if $\lim_{n \to \infty} \| \xi_n - \varphi \| = 0$ and therefore we have

$$s - \liminf_{n \to \infty} F_n = G.$$

Further we get from the Mosco-convergence of $\{G_n : n \in \mathbb{N}\}$ that for every $\varphi \in G$ the relation

$$\sum_{k=0}^{n} b_{n,k} \langle \varphi_k, x^* \rangle \xrightarrow{w} \langle \xi, x^* \rangle \ (x^* \in \mathcal{X}^*)$$

holds. Because of $\langle \varphi_k, x^* \rangle \in \mathbb{R}$ $(\varphi_k \in \mathcal{X},\ x^* \in \mathcal{X}^*)$ and the convergence equivalence of B we get that the real sequence $\left\{ \sum_{k=1}^{n} b_{n,k} \langle \varphi_k, x^* \rangle : n \in \mathbb{N} \right\}$ converges in the Banach-space $(\mathbb{R}, |\cdot|)$ if and only if $\{\langle \varphi_n, x^* \rangle : n \in \mathbb{N}\}$ is convergent to the same limit. Therefore we get

$$w - \liminf_{n \to \infty} F_n = G.$$

If we put everything together we have shown, that $F_n \to G$ a.s. in the Mosco-sense.

5. Applications

In the following we mention some applications for the given result. In many applications (e.g. economy) the assumption of a martingale or submartingale is often too strict to consider all relevant facts of the problem.

(1) Earlier papers have enhanced the interest in the use of a martingale-type approach to the study of the convergence properties, in a stochastic environment, of recursive identification and control schemes (see Ljung[9], Goodwin *et al.*[4], Solo[12], Sin and Goodwin[11] and Landau[8]).

(2) In economy capital market models are based on martingales-type approaches (see e.g. many papers of Föllmer). Let F_n be the set of all possible quotations of a stock at time n and $\mathcal{A}_n$ be the information about the company at time n. Then the expected set of quotations of the given stock at time $n + 1$ given the information of the company up to the time n can be regarded as a weighted sum of quotations at all earlier times. If we only consider the time n (martingale) we have the problem of seasonal deviations.

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