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ON THE FRACTIONAL CALCULUS OF A GENERAL CLASS OF POLYNOMIALS

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The present paper aims at the derivation of certain fractional calculus formula, which involves a general class of polynomials. Relevance with certain known results is also pointed out.

1. INTRODUCTION

Vyas and Banerji[9] obtained a fractional integral formula for the function $(\alpha z+\beta)^a$ using generalized beta integral, while Ross[2] obtained the fractional integral formula for the same function using series expansion method. Very recently Srivastava and Nishimoto[7] observed that the fractional integral formula, considered by Vyas and Banerji[9], and Ross[2], follows readily from, and is no more general than, the well known (classical) result (2.5) of this article. The aim of the present paper is to obtain the fractional calculus formula, using series expansion method, for the general class of polynomials which were introduced by Srivastava[4]. The name general class of polynomials, itself indicates the importance of the results, because we can derive a number of fractional calculus formulae for various classical orthogonal polynomials.

The Riemann-Liouville operator of fractional calculus of order α is

$$ {}_cR_x^{-\alpha} = {}_cD_x^{\alpha}[f(x)] = \frac{1}{\Gamma(-\alpha)} \int_c^x (x-t)^{-\alpha-1} f(t)\,dt, \ \mathrm{Re}\ (\alpha) > 0 \quad \ldots (1.1)$$

$$ = \frac{d^m}{dx^m}\, {}_cD_x^{\alpha-m}[f(x)],\ 0 \le \mathrm{Re}(\alpha) < m \quad \ldots (1.2)$$

where $f(x)$ is a locally integrable function and m is a positive integer.

For convenience, we take $c = 0$ in (1.1) to replace ${}_cD_x^{\alpha}$ by ${}_0D_x^{\alpha}$ or simply by D_x^{α}.

The general class of polynomials, $S_n^m(x)$, introduced by Srivastava[4] is as follows :

$$S_n^m(x) = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} x^j \qquad \text{... (1.3)}$$

where m is an arbitrary positive integer and the coefficient $A_{n,j}$ $(n, j \geq 0)$ are arbitrary constants, real or complex.

By suitably specializing the coefficients $A_{n,j}$, the polynomial set $S_n^m(x)$ reduces to the various classical orthogonal polynomial (Szegö[8]). The particular cases of our main results have come out in terms of the well-known Kampé de Fériet double hypergeometric functions (cf. Srivastava and Karlsson[6], p. 27) :

$$F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x, y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\frac{y^s}{s!}. \qquad \text{... (1.4)}$$

The convergence conditions for which are

(i) $p + q < l + m + 1$, $p + k < l + n + 1$, $|x| < \infty$ and $|y| < \infty$, or

(ii) $p + q = l + m + 1$, $p + k = l + n + 1$ and

$|x|^{1/(p-l)} + |y|^{1/(p-l)} < 1$, if $p > l$

$\max\{|x|, |y|\} < 1$ if $p \leq l$.

2. The Fractional Calculus Formula

In what follows now, we prove the following fractional calculus formulae :

$$D_x^{\mu}[x^k (x+\xi)^{-\lambda} S_n^m(x^{\rho}(x+\xi)^{-\sigma})]$$

$$= \xi^{-\lambda} x^{k-\mu} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}\, \Gamma\left[\begin{matrix}\rho j + k + 1 \\ \rho j + k - \mu + 1\end{matrix}\right]$$

$$\times \left(\frac{x^{\rho}}{\xi^{\sigma}}\right)^j {}_2F_1\left[\begin{matrix}\lambda + \sigma j,\ k + \rho j + 1\,; \\ k + \rho j - \mu + 1 \quad ;\end{matrix}\; -\frac{x}{\xi}\right], \qquad \text{... (2.1)}$$

valid for $\min(k, \lambda, \rho, \sigma) > 0$, $|x/\xi| < 1$ and $\mathrm{Re}(k + \rho j - \mu + 1) > 0$ and

$$D_x^{\mu}[x^k (x+\xi)^{-\lambda} S_n^m (x^{\rho}(x+\xi)^{-\sigma}) S_p^q (x^{\delta})]$$

$$= \xi^{-\lambda} x^{k-\mu} \sum_{j=0}^{[n/m]} \sum_{i=0}^{[p/q]} \frac{(-n)_{mj}(-p)_{qi}}{i!\, j!} A_{n,j} A_{p,i}$$

$$\times \Gamma\left[\begin{matrix} \rho j + \delta i + k + 1 \\ \rho j + \delta i + k - \mu + 1 \end{matrix}\right] x^{\rho j + \delta i} \xi^{-\sigma j} \times {}_2F_1\left[\begin{matrix} \lambda + \sigma j,\ k + \rho j + \delta i + 1\ ; \\ k + \rho j + \delta i - \mu + 1\ ; \end{matrix} \ -\frac{x}{\xi}\right], \quad \text{... (2.2)}$$

valid for min $(k, \lambda, \rho, \sigma, \delta) > 0$, $|x/\xi| < 1$, Re $(k + \rho j + \delta i - \mu + 1) > 0$, where ${}_2F_1$ $(\cdot)$ is the well known Gauss' function, S_n^m $(\cdot)$ is defined in (1.3) and

$$\Gamma\left[\begin{matrix} a_p \\ b_p \end{matrix}\right] = \frac{\Gamma(a_1) \ldots \Gamma(a_p)}{\Gamma(b_1) \ldots \Gamma(b_q)} . \quad \text{... (2.3)}$$

It may be noted that the parameters α and β used in (1.4), are real or complex.

PROOFS : To prove the results (2.1) and (2.2), we first use the series representation (1.3) and then expand the binomial terms like

$$(x+\xi)^{-\lambda} = \xi^{-\lambda} \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \left(-\frac{x}{\xi}\right)^m, \quad \text{... (2.4)}$$

provided $|x/\xi| < 1$. Making use of the well known formula as follows, or by using (1.1), taking $f(x) = x^{\lambda}$, we have

$$D_x^{\mu}(x^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} x^{\lambda-\mu}, \text{ Re } (\lambda) > -1. \quad \text{... (2.5)}$$

Further using the series expansion of ${}_2F_1$ $(\cdot)$ during the course of analysis, we arrive at the required results.

3. APPLICATIONS

As the special cases of our main results, if we take $\sigma = 0$ and $\lambda = 0$, we deduce the results earlier given by Saigo and Raina[3].

By taking $\sigma = 0$, fractional calculus formula (2.1) readily yields

$$D_x^{\mu}[x^k (x+\xi)^{-\lambda} S_n^m (x^{\rho})] = \xi^{-\lambda} x^{k-\mu} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}$$

$$\times \Gamma\left[\begin{matrix} \rho j + k + 1 \\ \rho j + k - \mu + 1 \end{matrix}\right] x^{\rho j} {}_2F_1\left[\begin{matrix} \lambda, k + \rho j + 1\ ; \\ k + \rho j - \mu + 1\ ; \end{matrix} \ -\frac{x}{\xi}\right], \quad \text{... (3.1)}$$

while (2.2) reduces to

$$D_x^{\mu}\left[x^k (x+\xi)^{-\lambda} S_n^m (x^{\rho}) S_p^q (x^{\delta})\right]$$

$$= \xi^{-\lambda} x^{k-\mu} \sum_{j=0}^{[n/m]} \sum_{i=0}^{[p/q]} \frac{(-n)_{mj}(-p)_{qi}}{i!\, j!} A_{n,j} A_{p,i}$$

$$\times \Gamma\begin{bmatrix} \rho j+\delta i+k+1 \\ \rho j+\delta i+k-\mu+1 \end{bmatrix} x^{\rho j+\delta i}\, {}_2F_1\begin{bmatrix} \lambda, k+\rho j+\delta i+1 \ ; & -\dfrac{x}{\xi} \\ k+\rho j+\delta i-\mu+1 \ ; & \end{bmatrix}.$$

... (3.2)

If we take $\lambda = 0$ in (3.1) and (3.2), these formulae reduce to

$$D_x^{\mu}\left[x^k S_n^m (x^{\rho})\right] = x^{k-\mu} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}$$

$$\times \Gamma\begin{bmatrix} \rho j+k+1 \\ \rho j+k-\mu+1 \end{bmatrix} x^{\rho j}, \qquad \text{... (3.3)}$$

and

$$D_x^{\mu}\left[x^k S_n^m (x^{\rho}) S_p^q (x^{\delta})\right] = x^{k-\mu} \sum_{j=0}^{[n/m]} \sum_{i=0}^{[q/p]} \frac{(-n)_{mj}(-p)_{qi}}{i!\, j!} A_{n,j} A_{p,i}$$

$$\times \Gamma\begin{bmatrix} \rho j+\delta i+k+1 \\ \rho j+\delta i+k-\mu+1 \end{bmatrix} x^{\rho j+\delta i}. \qquad \text{... (3.4)}$$

Out of these two formulae, (3.3), on specialization of $A_{n,j}$, reduces to a very special type of polynomial, earlier suggested by Srivastava and Buschman[5] (p. 361). Also, the representation (3.3) was used by Vyas *et al.*[10].

Particular Cases

In what follows now, are few particular cases of interest.

(i) By setting $m = \rho = \sigma = 1$, $k = \alpha$ and $m = \rho = 1$ and $\sigma = 0$, and

$$A_{n,j} = \frac{(1+\alpha+\beta+n)_j (1+\alpha)_n}{(1+\alpha)_j\, n!}$$

in (2.1) and (3.1), we obtain results involving Jacobi polynomial $P_n^{(\alpha,\beta)}$ and Kampé de Fériet double hypergeometric function, given in (1.4).

(ii) By putting $m = \rho = 1$, $k = \alpha$ and

$$A_{n,j} = \frac{\Gamma(1+\alpha+n)}{n!\,\Gamma(1+\alpha+j)} \text{ in (3.3),}$$

we get (Erdélyi *et al.*[1] p. 190) as the special case, and under same substitution and

$$A_{n,j} = \frac{\Gamma(1+\alpha+n)\,\Gamma(1+n+\alpha+\beta)}{n!\,\Gamma(1+\alpha+j)} ,$$

the same equation yields a result involving Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha-\mu,\beta+\mu)}$.

(iii) For $\rho = \delta = m = q = 1$, $k = \alpha$ and

$$A_{n,j} = \frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+j)\,n!}$$

and

$$A_{p,i} = \frac{\Gamma(1+p+\beta)}{p!\,\Gamma(1+i+\beta)} ,$$

(3.4) yields a result involving the Kampé de Fériet function and $L_p^{(\beta)}$.

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