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A NEW CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS WITH TWO FIXED POINTS

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This paper deals with functions of the form $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$. We examine the subclass for which $(1-\mu)f(z_0)/z_0 + \mu f'(z_0) = 1$ $(-1 < z_0 < 1)$ with reference to fractional derivatives. We have obtained coefficient bounds, distortion theorem and a few characterisation theorems.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

which are univalent in the unit disk $D = \{z : |z| < 1\}$.

Silverman[1] studied the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \qquad \dots (1.1)$$

where, either

$$f(z_0) = z_0 \qquad (-1 < z_0 < 1\,;\; z_0 \neq 0)$$

or

$$f'(z_0) = 1 \qquad (-1 < z_0 < 1).$$

Further Uralegaddi and Somanatha[3] generalized this class by considering the functions of the form defined by (1.1) with

$$(1-\mu)f(z_0)/z_0+\mu f'(z_0) = 1 \qquad \text{... (1.2)}$$

where

$$-1<z_0<1,\quad 0\le\mu\le 1.$$

Definition 1 — A function $f(z)$ is said to be convex of order α, if

$$\mathrm{Re}\{1+zf''(z)/f'(z)\} > \alpha\ (z\in D: 0\le\alpha<1). \qquad \text{... (1.3)}$$

Definition 2 — The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{(-\delta)}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z \frac{f(z)}{(z-\theta)^{(1-\delta)}}\,d\theta \qquad \text{... (1.4)}$$

where $\delta>0$, $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\theta)^{\delta-1}$ is removed by requiring $\log(z-\theta)$ to be real when $(z-\theta)>0$.

Definition 3 — The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(z)}{(z-\theta)^{\delta}}\,d\theta \qquad \text{... (1.5)}$$

where $0\le\delta<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\theta)^{-\delta}$ is removed as in Definition 2 above.

Definition 4 — A function $f(z)$ defined by (1.1) and satisfying (1.2) is said to be in the class $\rho(\mu,\delta,A,B,z_0)$ if

$$\Gamma(2-\delta)\,z^{\delta-1}D_z^{\delta}f(z)=a_1\frac{1+Aw(z)}{1+Bw(z)}$$

where $w(z)$ is analytic in D with $w(0)=0$, $|w(z)|<1$ and $-1\le B<A\le 1$.

2. Coefficient Inequalities

Theorem 1 — The function $f(z)$ defined by (1.1) belongs to $\rho(\mu,\delta,A,B,z_0)$ if and if only if

$$\sum_{n=2}^{\infty}\left\{\frac{(1-B)}{(A-B)}\phi(\delta,n)-[(1-\mu)+n\mu]\,z_0^{n-1}\right\}a_n\le 1 \qquad \text{... (2.1)}$$

where

$$\phi(\delta,n)=\frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)}.$$

PROOF : Suppose that $f(z)$ belongs to $\rho(\mu, \delta, A, B, z_0)$, then we have

$$F(z) = a_1 \frac{1 + Aw(z)}{1 + Bw(z)} \qquad (-1 \leq B < A \leq 1) \qquad \ldots (2.2)$$

where $w(z)$ is analytic in D with $w(0) = 0$, $| w(z) | < 1$ and

$$F(z) = \Gamma(2-\delta)\, z^{\delta-1} D_z^{\delta} f(z) = a_1 - \sum_{n=2}^{\infty} \phi(\delta, n)\, a_n z^{n-1},$$

and

$$\phi(\delta, n) = \frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)} .$$

Equation (2.2) is equivalent to

$$\left| \frac{F(z) - a_1}{BF(z) - a_1 A} \right| = | w(z) | < 1. \qquad \ldots (2.3)$$

Since $| \text{Re}\ (z) | \leq | z |$ for any z, we have from (2.3)

$$\text{Re} \left\{ \sum_{n=2}^{\infty} \phi(\delta, n)\, a_n z^{n-1} \Big/ \left[(A-B)a_1 + \sum_{n=2}^{\infty} B\phi(\delta, n)\, a_n z^n \right] \right\} \leq 1. \qquad \ldots (2.4)$$

Choose values of z on the real axis so that $F(z)$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1$ through the real values, we get

$$\sum_{n=2}^{\infty} (1-B)\phi(\delta, n) a_n \leq a_n (A-B). \qquad \ldots (2.5)$$

Finally substituting $a_1 = 1 + \sum_{n=2}^{\infty} [(1-\mu) + n\mu]\, a_n z_0^{n-1}$ in (2.5), we get (2.1).

Conversely suppose that (2.1) holds. Consider

$$| F(z) - a_1 | - | BF(z) - a_1 A |$$

$$= \left| \sum_{n=2}^{\infty} \phi(\delta, n)\, a_n z^{n-1} \right| - \left| (A-B)a_1 + \sum_{n=2}^{\infty} B\phi(\delta, n)\, a_n z^{n-1} \right|$$

$$\leq \sum_{n=2}^{\infty} (1-B)\, \phi(\delta, n)\, a_n - a_1 (A-B)$$

$$\leq 0, \quad \text{by hypothesis.}$$

Hence, by maximum modulus theorem, we get

$$\left| \frac{F(z) - a_1}{BF(z) - a_1 A} \right| < 1 \text{ for all } z \in D,$$

which consequently implies that there exists an analytic function $w(z)$ such that $w(0) = 0$ and $|w(z)| < 1$ and that

$$\frac{F(z) - a_1}{BF(z) - a_1 A} = w(z)$$

which in turn implies that $f(z)$ belongs to $\rho(\mu, \delta, A, B, z_0)$.

Corollary — Let the function $f(z)$ defined by (1.1) belong to $\rho(\mu, \delta, A, B, z_0)$. Then

$$a_n \leq \{[(1-B)/(A-B)]\, \phi(\delta, n) - [(1-\mu) + n\mu]\, z_0^{n-1}\}^{-1} \quad (n \geq 2)$$

with equality for

$$f(z) = \frac{(1-B)\, \phi\, (\delta, n) z - (A-B) z^n}{\left\{ (1-B)\, \phi(\delta, n) - [(1-\mu) + n\mu]\, (A-B)\, z_0^{n-1} \right\}}\,. \quad \ldots (2.6)$$

3. A Distortion Theorem

Theorem 2 — Let the function $f(z)$ defined by (1.1) belong to $\rho(\mu, \delta, A, B, z_0)$. Then

$$|f(z)| \geq a_1 \left(|z| - \frac{(A-B)(2-\delta)}{2(1-B)} |z|^2 \right) \quad \ldots (3.1)$$

and

$$|f(z)| \leq a_1 \left(|z| - \frac{(A-B)(2-\delta)}{2(1-B)} |z|^2 \right) \quad \ldots (3.2)$$

for $z \in D$. Also

$$|D_z^{\delta} f(z)| \geq \frac{a_1}{\Gamma(2-\delta)} \left(|z|^{1-\delta} - \frac{(A-B)}{(1-B)} |z|^{2-\delta} \right) \quad \ldots (3.3)$$

and

$$|D_z^{\delta} f(z)| \geq \frac{a_1}{\Gamma(2-\delta)} \left(|z|^{1-\delta} + \frac{(A-B)}{(1-B)} |z|^{2-\delta} \right) \quad \ldots (3.4)$$

whenever $z \in D$.

Proof : In view of eqn. (2.5) and the fact that $\phi(\delta, n)$ is non-decreasing for $n \geq 2$, we have

$$\frac{2(1-B)}{(2-\delta)} \sum_{n=2}^{\infty} a_1 \leq \sum_{n=2}^{\infty} \phi\, (\delta, n)\, (1-B)\, a_n \leq a_1\, (A-B) \quad \ldots (3.5)$$

which is equivalent to

$$\sum_{n=2}^{\infty} a_n \le \frac{a_1 (A-B)(2-\delta)}{2(1-B)}. \qquad \ldots (3.6)$$

Consequently, we obtain

$$|f(z)| \ge a_1 |z| - |z|^2 \sum_{n=2}^{\infty} a_n \ge a_1 \left(|z| - \frac{(A-B)(2-\delta)}{2(1-B)} |z|^2 \right) \qquad \ldots (3.7)$$

and

$$|f(z)| \le a_1 |z| + |z|^2 \sum_{n=2}^{\infty} a_n \le a_1 \left(|z| + \frac{(A-B)(2-\delta)}{2(1-B)} |z|^2 \right). \qquad \ldots (3.8)$$

Further, by using second inequality in (3.5), we observe that

$$|\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)| \ge a_1 |z| - \sum_{n=2}^{\infty} \phi(\delta, n) a_n |z|^n$$

$$\ge a_1 |z| - |z|^2 \left(\sum_{n=2}^{\infty} \phi(\delta, n) a_n \right)$$

$$\ge a_1 (|z| - [(A-B)/(1-B)] |z|^2)$$

which is equivalent to (3.3).

Finally

$$|\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)| \le a_1 |z| + \sum_{n=2}^{\infty} \phi(\delta, n) a_n |z|^n$$

$$\le a_1 |z| + |z|^2 \left(\sum_{n=2}^{\infty} \phi(\delta, n) a_n \right)$$

$$\le a_1 (|z| + [(A-B)/(1-B)] |z|^2)$$

which yields (3.4). Hence the proof is complete.

Corollary — Let the function $f(z)$ defined by (1.1) belong to $\rho(\mu, \delta, A, B, z_0)$. Then $f(z)$ is included in a disk with its centre at origin and radius r given by

$$r = a_1 (1 + [(A-B)(2-\delta)/2(1-B)])$$

and $D_z^{\delta} f(z)$ is included in a disk with its centre at the origin and radius $\mathcal{R}$ given by

$$\mathcal{R} = \frac{a_1}{\Gamma(2-\delta)} \{1 + [(A-B)/(1-B)]\}.$$

4. Radius of Convexity

Theorem 4 — Let the function $f(z)$ defined by (1.1) belong to $\rho(\mu, \delta, A, B, z_0)$. Then $f(z)$ is convex in the disk

$$|z| < r = r(\delta, A, B) = \inf_n \left[\frac{(1-B)\,\phi(\delta, n)}{n^2 (A-B)} \right]^{(1/[n-1])} \quad (n \geq 2). \qquad \text{... (4.1)}$$

The result is sharp for the function given by (2.6).

Proof : To prove this result it is sufficient to prove that $|zf''(z)/f'(z)| \leq 1$ for $|z| < r(\delta, A, B)$.

A simple calculation gives us

$$|zf''(z)/f'(z)| \leq \left(\sum_{n=2}^{\infty} n(n-1)\, a_n |z|^{n-1} \right) \Big/ \left(a_1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \right).$$

Clearly, $|zf''(z)/f'(z)| \leq 1$ if

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n |z|^{n-1} \right) \leq \left(a_1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \right). \qquad \text{... (4.2)}$$

Using $a_1 = 1 + \sum_{n=2}^{\infty} a_n [(1-\mu) + n\mu]\, z_0^{n-1}$ in (3.1), we are lead to

$$\sum_{n=2}^{\infty} a_n \; \{n^2 |z|^{n-1} - [(1-\mu) + n\mu]\, z_0^{n-1}\} \leq 1. \qquad \text{... (4.3)}$$

By Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \left\{ \frac{(1-B)}{(A-B)}\, \phi(\delta, n) - [(1-\mu) + n\mu]\, z_0^{n-1} \right\} \leq 1.$$

Hence (4.3) will hold, if

$$n^2 |z|^{n-1} - [(1-\mu) + n\mu]\, z_0^{n-1} \leq \frac{(1-B)}{(A-B)}\, \phi(\delta, n) - [(1-\mu) + n\mu]\, z_0^{n-1}$$

or equivalently

$$|z|^{n-1} \leq \frac{(1-B)}{n^2 (A-B)}\, \phi(\delta, n)$$

which in turn implies the assertion of the theorem.

5. Further Properties of the Class $\rho(\mu, \delta, A, B, z_0)$

Theorem 5 — Let $0 \le \delta \le 1$, $0 \le \mu \le 1$, $-1 \le B < A \le 1$, $-1 \le B' < A' \le 1$. Then

$$\rho(\mu, \delta, A, B, z_0) = \rho(\mu, \delta, A', B', z_0) \qquad \text{... (5.1)}$$

if and only if

$$\frac{(A-B)}{(1-B)} = \frac{(A'-B')}{(1-B')}. \qquad \text{... (5.2)}$$

Furthermore

$$\rho(\mu, \delta, A, B, z_0) = \rho(\mu, \delta, [2(A-B)/(1-B)]-1, -1, z_0). \qquad \text{... (5.3)}$$

Proof : Let $f(z) \in \rho(\mu, \delta, A, B, z_0)$ and (5.2) hold true. Then by Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \left\{ \frac{(1-B)}{(A-B)} \phi(\delta, n) - [(1-\mu) + n\mu] z_0^{n-1} \right\}$$

$$= \sum_{n=2} a_n \left\{ \frac{(1-B')}{(A'-B')} \phi(\delta, n) - [(1-\mu) + n\mu] z_0^{n-1} \right\} < 1$$

which implies that $f(z) \in \rho(\mu, \delta, A', B', z_0)$.

Similarly it can be shown that $f(z) \in \rho(\mu, \delta, A', B', z_0)$ implies $f(z) \in \rho(\mu, \delta, A, B, z_0)$ under the condition (5.2). Hence (5.2) implies (5.1).

Conversely suppose that (5.1) holds true. We note that the function defined by (1.1) belonging to $\rho(\mu, \delta, A, B, z_0)$ will belong to $\rho(\mu, \delta, A', B', z_0)$ only if

$$\sum_{n=2}^{\infty} a_n \left\{ \frac{(1-B)}{(A-B)} \phi(\delta, n) - [(1-\mu) + n\mu] z_0^{n-1} \right\}$$

$$\le \sum_{n=2}^{\infty} a_n \left\{ \frac{(1-B')}{(A'-B')} \phi(\delta, n) - [(1-\mu) + n\mu] z_0^{n-1} \right\}$$

which is equivalent to

$$\frac{(A-B)}{(1-B)} = \frac{(A'-B')}{(1-B')} \qquad \text{... (5.4)}$$

with similar arguments, we can deduce

$$\frac{(A'-B')}{(1-B')} = \frac{(A-B)}{(1-B)}. \qquad \text{... (5.5)}$$

Inequalities (5.4) and (5.5) together yield (5.2). Finally, for $B' = -1$, we get (5.3).

Theorem 5 — Let $0 \le \delta \le 1$, $-1 \le B < A_1 \le A_2 \le 1$. Then

$$\rho(\mu, \delta, A_1, B, z_0) \subseteq \rho(\mu, \delta, A_2, B, z_0). \quad \text{... (5.6)}$$

The proof of Theorem 5 is a simple application of Theorem 1.

Theorem 6 — Let $0 \leq \delta \leq 1$, $-1 \leq B_1 \leq B_2 < A \leq 1$. Then

$$\rho(\mu, \delta, A, B_1, z_0) \supseteq \rho(\mu, \delta, A, B_2, z_0). \quad \text{... (5.7)}$$

PROOF : By using Theorem 4, we have

$$\rho(\mu, \delta, A, B_1, z_0) = \rho(\mu, \delta, [2(A-B_1)/(1-B_1)]-1, -1, z_0)$$

and

$$\rho(\mu, \delta, A, B_2, z_0) = \rho(\mu, \delta, [2(A-B_2)/(1-B_2)]-1, -1, z_0).$$

And noting that

$$0 \leq [2(A-B_2)/(1-B_2)]-1 \leq [2(A-B_1)/(1-B_1)]-1 \leq 1, \quad \text{... (5.8)}$$

Theorem 6 leads us to (5.7).

Corollary — Let $0 \leq \delta \leq 1$, $-1 \leq B_1 \leq B_2 < A_1 \leq A_2 \leq 1$. Then

$$\rho(\mu, \delta, A_1, B_2, z_0) \subseteq \rho(\mu, \delta, A_1, B_1, z_0) \subseteq \rho(\mu, \delta, A_2, B_1, z_0).$$

Now we state two theorems which can be easily obtained respectively on the lines of proofs of Theorem 6 and Theorem 7 in Srivastava and Owa[2].

Theorem 7 — Let $0 \leq \delta_1 \leq \delta_2 \leq 1$, $-1 \leq B < A \leq 1$. Then

$$\rho(\mu, \delta_1, A, B, z_0) \supseteq \rho(\mu, \delta_2, A, B, z_0).$$

Theorem 8 — Let $0 \leq \delta \leq 1$, $-1 \leq B < A \leq 1$. Then

$$\rho(\mu, 1, A, B, z_0) \subseteq \rho(\mu, \delta, A, B, z_0).$$

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