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ON SIMILARITY SOLUTIONS FOR TYPE D SPHERICAL AND PSEUDO-SPHERICAL FLUID DISTRIBUTIONS IN 5-FLAT FORM IN GENERAL RELATIVITY

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Using the Lie group of transformations techniques all the similarity type D solutions for spherical and pseudo-spherical perfect fluid distributions of embedding class one have been derived by considering the 5-flat space. The solutions so obtained have been seen to describe Zeldovich fluids. The solutions being expressible in 5-flat form may be quite useful to those who are linking the embedding space to the internal symmetries of elementary particle physics. Also the maximal analytic extension for the solutions can be had through an isometric embedding into flat space.

1. INTRODUCTION

This is well-known that all the type D perfect fluid distributions of class one with non-vanishing acceleration are expressible by $G_3(2, S)$ metric[1]. Some of the solutions belonging to the said category are due to Kohler Chao[2], Barnes[1], Gupta *et al.*[3, 4]. In the present article the authors have tried to tackle the problems for spherical and pseudo-spherical cases using the Lie point group of transformations[5]. The solutions so obtained are seen to describe Zeldovich fluids[3]. It is worth pointing out here that all the Zeldovich fluids of above class are known in 4 dimensional normal form[3]. The crux of the problem lies in the fact that the solutions are expressible in 5-flat form explicitly.

2. BASIC METRICS

An appropriate metric $G_3(2, S)$ admitting a three-parameters group of isometries with two dimensional trajectories r = constant and t = constant can be expressed as

$$ds^2 = -A(r, t)\, dr^2 - B(r, t)\, [d\theta^2 + f^2(\theta)\, d\phi^2] + C(r, t)\, dt^2 + 2D(r, t)\, dr\, dt \quad \dots (1)$$

where f is sin θ, sinh θ and θ representing spherical, pseudo-spherical (hyperbolic) and plane symmetric cases.

The metric (1) is said to be of embedding class one if it can be transformed to the metric

$$ds^2 = d\sigma^2 + edu^2 \quad \text{... (2)}$$

where $e = \pm 1, u = u(r, t)$ and $d\sigma^2$ represents a four dimensional flat metric.

The metrics associated with the spherical, hyperbolic and plane symmetries can be expressed as

$$ds^2 = -dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2 + edu^2 \quad \text{... (3)}$$

$$ds^2 = -dr^2 - t^2 (d\theta^2 + \sin^2\theta\, d\phi^2) - dt^2 + du^2 \quad \text{... (4)}$$

$$ds^2 = -dr^2 - r^2 (d\theta^2 + \sinh^2\theta\, d\phi^2) + dt^2 - du^2 \quad \text{... (5)}$$

$$ds^2 = -dr^2 - t^2 (d\theta^2 + \sinh^2\theta\, d\phi^2) + dt^2 + edu^2 \quad \text{... (6)}$$

$$ds^2 = -dr^2 - (t \mp r)^2 (d\theta^2 + \theta^2 d\phi^2) + dt^2 + edu^2. \quad \text{... (7)}$$

The metrics (3)-(7) can also be expressed by means of a single metric

$$ds^2 = -Adr^2 - (mr + nt)^2 [d\theta^2 + f^2(\theta)\, d\phi^2] + Cdt^2 + Ddu^2 \quad \text{... (8)}$$

where A, C and D are constants of unit modulus and $\frac{Cm^2 - An^2}{AC} = 1, -1, 0$ for $f(\theta) = \sin\theta$, sinh θ and θ respectively. However, the metrics (3) to (7) can be reobtained from (8) by setting the following values of m, n, A, C and D

metric (3) : $m = 1,\ n = 0,\ A = C = 1,\ D = \pm 1$ ... (9)

metric (4) : $m = 0,\ n = 1,\ A = 1,\ C = -1,\ D = 1$ ... (10)

metric (5) : $m = 1,\ n = 0,\ A = -1,\ C = 1,\ D = -1$ ... (11)

metric (6) : $m = 0,\ n = 1,\ A = 1,\ C = 1,\ D = \pm 1$ ... (12)

metric (7) : $m = \mp n = 1,\ A = 1,\ C = 1,\ D = \pm 1.$ ... (13)

The type D perfect fluid distributions of embedding class one can be obtained by[4]

$$\frac{R_{2323}}{B^2 f^2(\theta)} = T_2^2. \quad \text{... (14)}$$

The relation (14) together with (8) gives rise the differential equation

$$AC(\ddot{u}u'' - \dot{u}'^2) + \frac{(mCu' - nA\dot{u})}{(mr+nt)} [(A - Du'^2)\,\ddot{u} + 2D\dot{u}u'\,\dot{u}' - (C + D\,\dot{u}^2)\,u'']$$

$$- \frac{(mCu' - nA\dot{u}\,)^2}{AC\,(mr+nt)^2} [AC + AD\dot{u}^2 - CDu'^2] = 0$$

$$\equiv H(r, t, \dot{u}, u', \dot{u}', u'', \ddot{u}) \text{ (let)} \qquad \ldots (15)$$

and the expressions for pressure and density are

$$8\pi p = -ACD\,(mCu' - nA\dot{u}\,)^2/(mr+nt)^2\,(AC + AD\dot{u}^2 - CDu'^2) \qquad \ldots (15a)$$

$$8\pi\rho = 8\pi p - 2ACD\,(\ddot{u}\,u'' - \dot{u}'^2)/(AC + AD\,\dot{u}^2 - CDu'^2)^2. \qquad \ldots (15b)$$

3. Invariance Transformations

In order to search for a mapping under which eqn. (15) is invariant we shall use the well-known method of Lie point group of transformations[5] according to which the invariance condition reads as

$$R\frac{\partial H}{\partial r} + T\frac{\partial H}{\partial t} + U\frac{\partial H}{\partial u} + [U_r]\frac{\partial H}{\partial u_r} + [U_t]\frac{\partial H}{\partial u_t} + [U_{rt}]\frac{\partial H}{\partial u_{rt}}$$

$$+ [U_{rr}]\frac{\partial H}{\partial u_{rr}} + [U_{tt}]\frac{\partial H}{\partial u_{tt}} = 0 \qquad \ldots (16)$$

on inserting (15) into (16) and equating the various derivatives of u to zero we obtain the following set of equations :

(i) For the metrics (3) and (5) setting $C = m = 1$, $n = 0$

$$T_r = U_t = U_r = T_u = R_u = U_{uu} = 0 \qquad \ldots (17)$$

$$4(U_u - R_r) + Ar\,R_{tt} = 0 \qquad \ldots (18)$$

$$\left.\begin{aligned}
&R + r(U_u - R_r - T_t) = 0\\
&R + r(4U_u - 3R_r - 2T_t) = 0\\
&2(2U_u - R_r - T_t) - r\,R_{rr} = 0\\
&2R_t + r(T_{tt} - 2R_{rt}) = 0\\
&-2U_u + 2R_r + r\,R_{rr} - Ar\,R_{tt} = 0\\
&-R_t + r\,R_{rt} = 0\\
&(R + 2rU_u - 3rR_r) + Ar^2 R_{tt} = 0\\
&R + 2r\,U_u - r\,R_r - 2r\,T_t - r^2 R_{rr} = 0.
\end{aligned}\right| \qquad \ldots (19)$$

Solving the set (17)-(19), we get

$$U = \alpha u + \alpha_1, \quad R = \alpha r, \quad T = \alpha t + \alpha_2. \qquad \text{... (20)}$$

To get the similarity variable, we have the equations

$$\frac{dt}{T} = \frac{dr}{R} = \frac{du}{U}$$

which from (20) takes the form

$$\frac{dt}{\alpha t + \alpha_2} = \frac{dr}{\alpha r} = \frac{du}{\alpha u + \alpha_1} \qquad \text{... (21)}$$

which provide on integration

$$u = \frac{1}{\alpha}[r f(\eta) - \alpha_1] \qquad \text{... (22)}$$

where $\eta = \frac{\alpha t + \alpha_2}{r}$ the similarity variable, u given by (22) is the solution of (15) provided f satisfies

$$(f - \eta \bar{f}) \Bigg[\bar{\bar{f}}\left(-\alpha + \frac{A}{\alpha}\eta^2\right) + \frac{AD}{\alpha} f^2 \bar{\bar{f}} + \frac{Af}{\alpha} - \frac{Df^3}{\alpha^3} + \frac{3Df^2\bar{f}\eta}{\alpha^3} - \frac{A\bar{f}\eta}{\alpha}$$
$$- \frac{D\bar{f}^3}{\alpha^3}(A\alpha^2\eta - \eta^3) - \frac{Df\bar{f}^2}{\alpha^3}(3\eta^2 - A\alpha^2) \Bigg] = 0 \qquad \text{... (23)}$$

where $\bar{f} = \frac{df}{d\eta}$.

(ii) Similarily for the metrics (4) and (6), we get

$$u = \frac{1}{\alpha}\left[tf\left(\frac{\alpha r + \alpha_2}{t}\right) - \alpha_1 \right] \qquad \text{... (24)}$$

and u satisfies (15), provided

$$(f - \eta \bar{f}) \Bigg[\bar{\bar{f}}\left(\alpha - \frac{C\eta^2}{\alpha}\right) + \frac{CD}{\alpha} f^2 \bar{\bar{f}} - \frac{Cf}{\alpha} - \frac{Df^3}{\alpha^3} + \frac{3Df^2\bar{f}\eta}{\alpha^3} + \frac{C\bar{f}\eta}{\alpha}$$
$$- \frac{D\bar{f}^3}{\alpha^3}(C\alpha^2\eta - \eta^3) - \frac{Df\bar{f}^2}{\alpha^3}(3\eta^2 - C\alpha^2) \Bigg] = 0 \qquad \text{... (25)}$$

where $\bar{f} = \frac{df}{d\eta}, \quad \eta = \frac{\alpha r + \alpha_2}{t}$.

The vanishing of first factor of eqns. (23) and (25) gives rise the flat space while the second factor admit the following first integrals owing to the similarity method for ordinary differential equations[5].

$$\frac{\eta\bar{f}-f}{\eta+Df\bar{f}}=\left[\frac{\alpha^2\{-\alpha^2+A(\eta^2+Df^2)\}}{D(\eta^2+Df^2)^2+D\alpha^4+k\alpha^2(\eta^2+Df^2)}\right]^{\frac{1}{2}} \quad \ldots (26)$$

and

$$\frac{\eta\bar{f}-f}{\eta-Df\bar{f}}=\left[\frac{\alpha^2\{-\alpha^2+C(\eta^2-Df^2)\}}{-D(\eta^2-Df^2)^2-D\alpha^4+k\alpha^2(\eta^2-Df^2)}\right]^{\frac{1}{2}} \quad \ldots (27)$$

where k being a constant. Differential equations (26) and (27) can easily be sent to variable separable form by means of the transformations of the forms, $\eta = r\cos\phi$, $f = r\sin\phi$, or $\eta = r\cosh\phi$, $f = r\sinh\phi$ depending upon the sign of D and the particular equation.

(22) provides the following expressions for pressure and density

$$8\pi p = 8\pi\rho = \frac{-D(\eta\bar{f}-f)^2}{r^2[\alpha^2+D\alpha^2\bar{f}^2-AD(\eta\bar{f}-f)^2]} \quad \ldots (28a)$$

where

$$\bar{f}=\frac{f[D(\eta^2+Df^2)^2+D\alpha^4+k\alpha^2(\eta^2+Df^2)]^{1/2}+\eta[\alpha^2\{-\alpha^2+A(\eta^2+Df^2)\}]^{1/2}}{-Df[\alpha^2\{-\alpha^2+A(\eta^2+Df^2)\}]^{1/2}+\eta[D(\eta^2+Df^2)^2+D\alpha^4+k\alpha^2(\eta^2+Df^2)]^{1/2}}$$

and (24) provides,

$$8\pi p = 8\pi\rho = \frac{-D(\eta\bar{f}-f)^2}{t^2[\alpha^2-D\alpha^2\bar{f}^2+CD(\eta\bar{f}-f)^2]} \quad \ldots (28b)$$

where

$$\bar{f}=\frac{f[-D(\eta^2-Df^2)^2-D\alpha^4+k\alpha^2(\eta^2-Df^2)]^{1/2}+\eta[\alpha^2\{-\alpha^2+C(\eta^2-Df^2)\}]^{1/2}}{Df[\alpha^2\{-\alpha^2+C(\eta^2-Df^2)\}]^{1/2}+\eta[-D(\eta^2-Df^2)^2-D\alpha^4+k\alpha^2(\eta^2-Df^2)]^{1/2}}.$$

(28a, b) represent a stiff-fluid[1].

The plane symmetric case i.e. $m = \mp n$ and $f(\theta) = \theta$ has been left here as it has completely different type of solutions and needs a separate article.

4. Some Additional Solutions

The expressions for u given by (22) and (24) are not valid for $\alpha = 0$. However, fresh calculation starting from (21) or its equivalent for metrics (4) and (6) yields the results :

(i) For the metrics (3) and (5), eqns. (21) suggest the following form of u (for $\alpha = 0$)

$$u=\frac{\alpha_1}{\alpha_2}t+\alpha_1 f(r) \quad \ldots (29)$$

which on substitution into (15) for $C = 1$, $m = 1$, $n = 0$ gives

$$-\frac{\alpha_1^2 f'}{r}\left[\left(1+\frac{D\alpha_1^2}{\alpha_2^2}\right)(rf''+f')-AD\alpha_1^2 f'^3\right] = 0. \qquad \ldots (30)$$

First integral of (30) reads as

$$f' = \frac{1}{\sqrt{K_1 r^2 + K}}\,, \qquad \ldots (31)$$

where $K = \frac{AD\alpha_1^2\alpha_2^2}{\alpha_2^2 + D\alpha_1^2}$ and K_1 is an arbitrary constant.

The case $f' = 0$ ultimately leads to flat space-time. Unfortunately (29) with (31) do not represent perfect fluid distributions as isotropic conditions $T_1^1 = T_2^2 = T_3^3$ are not satisfied which are necessary if $T_4^1 = 0$ which is so in this case. However, the necessary condition (15) for perfect fluid distributions is very much satisfied in this case.

(ii) For the metrics (4) and (6) the case $\alpha = 0$ yields the following form of u

$$u = \frac{\alpha_1}{\alpha_2} r + \alpha_1 f(t) \qquad \ldots (32)$$

(32) satisfies the perfect fluid equation (15) if

$$\dot{f} = \frac{1}{\sqrt{K_1 t^2 + K}}\,,$$

where $$K = \frac{CD\alpha_1^2\alpha_2^2}{D\alpha_1^2 - \alpha_2^2}\,. \qquad \ldots (33)$$

(32) with (33) provides the following expressions for density and pressure :

$$8\pi p = 8\pi\rho = \frac{D\alpha_1^2\alpha_2^2}{K_1(\alpha_2^2 - D\alpha_1^2)\,t^4}\,. \qquad \ldots (34)$$

(34) represents a stiff-fluid with acceleration zero[1].

5. Conclusions

The spherical and pseudo-spherical metrics have been considered in 5-flat form. And later on they are subjected to perfect fluid conditions. The nonlinear differential equation which is of Monge's form is solved to get all the similarity solutions. As a consequence it has been observed that the solutions so obtained describe stiff-fluid distributions i.e. with equation of state energy density equal to pressure, the self gravitating fluids[4]. The solutions so obtained may be helpful in the description of elementary particle physics[6] and also may be used for the maximal analytic extension[7].

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