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A NOTE ON A CLASS OF TRANSLATION PLANES OF SQUARE ORDER

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Narayana Rao *et al.*[1] have constructed a class of translation planes of order p^{2r} where p is a prime ≥ 7, r is an odd natural number and $p \neq \pm 1 \pmod{10}$ and have shown that the translation complement of any plane of the class fixes one ideal point and divides the rest of the ideal points into at least two orbits. In this note we establish that this construction also works for the prime $p = 3$. Further it has been shown that the translation complement of the planes of the class constructed in this note divides the distinguished points into exactly 3 orbits one of which consists of a single distinguished point if $r > 1$. In the case of $r = 1$, the plane is Desarguesian.

§1. Narayana Rao *et al.*[1] have constructed a class of translation planes of order p^{2r} where p is a prime in which 5 is a non-square and r is an odd natural number and $p^r \geq 7$, through a one spread set C defined by $C = \left\{ \begin{bmatrix} a & b \\ -5^{-1} b^5 & a + b^3 \end{bmatrix} \middle| \; a, b \in GF(p^r) \right\}$ and have established that the translation complement of any plane of this class fixes the distinguished point $V(\infty)$ and divides the rest of the distinguished points into at least two orbits. The aim of this note is to extend this construction to the prime $p = 3$. This has become possible since 5 is also a nonsquare in $GF(3)$ and proofs of most of the results given in Narayana Rao *et al.*[1] are valid for $p = 3$ however in some cases the fact that $q \not\equiv 0 \pmod 3$ if q is a prime power greater than or equal to 7 is used. We supply separate proofs in these cases keeping in view that the characteristic of the planes constructed in this note is 3.

§2. We now take the 1-spread set C defined by $C = \{M(a, b) | \; a, b \; GF(3^r)\}$ where $M(a, b) = \begin{bmatrix} a & b \\ b^5 & a + b^3 \end{bmatrix}$ and r is an odd natural number. Let π be the translation plane of order 3^{2r} associated with C. In order to follow this note one should refer to Narayana Rao *et al.*[1] constantly. It may be noted that if $r = 1$ then C is closed under matrix addition and multiplication and therefore defines the Desarguesian plane of order 3^2. From now onwards we assume that r is an odd natural number greater than one. It may be noted that all the assertions and remarks made in section 1 of Narayana Rao *et al.*[1] are valid for $p = 3$ as nowhere it is assumed that $p \not\equiv 0 \bmod 3$. In section 2 of Narayana Rao *et al.*[1] collineations α, β, γ and collineations group H are given. These are present also in the planes of oder 3^{2r}. Further the actions of the collineations α, β, γ in the plane of order 3^{2r} coincide with the patterns given in Narayana Rao *et al.*[1]. We now give some additional collineations of π not given in Narayana Rao *et al.*[1]. Let $\delta(c)$ be the mapping (Rao[2]) on C defined by

$$\delta(c) : M(a, b) \to \begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M(a, b) \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad c \neq 0, \ c \in GF(3^r).$$

From the relation

$$\begin{bmatrix} c & 0 \\ 0 & c^3 \end{bmatrix} M(a, b) \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix} = M(ac^3, bc),$$

we conclude that $\delta(c)$ induces a collineation fixing $V(\infty)$ and $V(M(0, 0))$ and moving $V(M(a, b))$ onto $V(M(ac^3, bc))$. Since $(3^r - 1, 3) = 1$ the collineations $\delta(c)$ as c ranges over all nonzero elements of $GF(3^r)$, act transitively on the sets of distinguished points

$$\{V(M(a, 0)) \mid a \in GF(3^r), \ a \neq 0\}$$

and

$$\{V(M(0, b)) \mid b \in GF(3^r), \ b \neq 0\}$$

separately.

The set $W = \{\delta(c) \mid c \in GF(3^r), \ c \neq 0\}$ forms a group and it is of order $(3^r - 1)$. Further the group $\langle H, W \rangle$ fixes $V(\infty)$ and acts transitively on the sets of distinguished points $\{V(M(a, 0)) \mid a \in GF(3^r)$ and $\{V(M(a, b)) \mid a, b \ GF(3^r), \ b \neq 0\}$ separately.

§3. We now list some results about the non-existence of certain types of collineations and refer the reader to Narayana Rao *et al.*[1] for proofs whenever they are valid for $p = 3$ and supply fresh proofs whenever they are not valid for $p = 3$.

Lemma 3.1 — There is no collineation of π which fixes $V(\infty)$ and moves $V(M(0, 0))$ onto $V(M(a, b))$, $b = 0$.

PROOF : Let ϕ be a collineation of π fixing $V(\infty)$ and mapping $V(M(0, 0))$ onto $V(M(a, b))$, $b = 0$. Since C has the property that $M(x, y) \in C$ implies $-M(x, y) \in C$, a necessary condition (Lemma 1.1, Narayana Rao *et al.*[1]) for the existence of ϕ is that

$$M(a, b) + M(c; d) \in C \text{ for each } M(c, d) \in C.$$

That is $\begin{bmatrix} a & b \\ b^5 & a + b^3 \end{bmatrix} + \begin{bmatrix} c & d \\ d^5 & c + d^3 \end{bmatrix} \in C.$

This in turn implies that $(b+d)^5 = b^5 + d^5$, which after simplification becomes

$$2bd\,(b-d)^2\,(b+d) = 0.$$

This forces $d = \pm b$ implying that this is not an identity in d. Hence the Lemma.

Lemma 3.2 — There is no collineation of π which interchanges $V(\infty)$ and $V(M(0, 0))$.

PROOF : Same as the proof of Lemma 3.2 of Narayana Rao *et al.*[1].

Lemma 3.2(a) — If $M(c, d) \in C$, and $a \in GF(3^r)$, $a \neq 0$, then $aM(c,d) \in C$ implies $a = \pm 1$.

PROOF : $aM(c, d) = a \begin{bmatrix} a & d \\ d^5 & c+d^3 \end{bmatrix} = \begin{bmatrix} ac & ad \\ ad^5 & ac+ad^3 \end{bmatrix} \in C.$

This implies that $ad^5 = (ad)^5$ and $ad^3 = (ad)^3$ implying $a^2 = 1$. Thus $a = \pm 1$ and so Lemma is proved.

Lemma 3.3 — There is no collineation of π which fixes $V(M(0, 0))$ and moves $V(\infty)$ onto $V(M(a, b))$, $(a, b) \neq (0,0)$.

PROOF : Let ϕ be a collineation which fixes $V(M(0, 0))$ and maps $V(\infty)$ onto $V(M(a, b))$. Since C has the property that $M(x,y) \in C$ implies $-M(x, y) \in C$, a necessary condition[1] for the existence of ϕ is that for each $M(c,d) \in C$, $[M^{-1}(a,b) + M^{-1}(c,d)]^{-1} \in C$.

Let $|M(a, b)| = K$. Then $M^{-1}(a,b) = \frac{1}{K} M(a+b^3, -b)$.

Taking $d = 0$, we get

$$[M^{-1}(a,b) + M^{-1}(\alpha, 0)]^{-1} \in C \text{ for all } \alpha \in GF(3^r).$$

Then substituting the value of $M^{-1}(a,b)$ we get

$$\left[\frac{1}{K} M(a+b^3, -b) + \alpha^{-1} I\right]^{-1} \in C. \text{ This implies that}$$

$$KM^{-1}(a + b^3 + K\alpha^{-1}, -b) \in C.$$

Let $|M(a+b^3+K\alpha^{-1}, -b)| = L$. A similar simplification as before yields that $KL^{-1} M(a + K\alpha, b) \in C$. This can happen only if $L = \pm K$, by Lemma 3.2(a).

This is a contradiction since $M(a + K\alpha, b)$ has more than two distinct values as α ranges over $GF(3^r)$. Therefore the Lemma is proved.

Lemma 3.4 — There is no collineation of π which sends $V(M(a, b))$ onto $V(\infty)$ and $V(\infty)$ onto $V(M(0, 0))$, $(a,b) \neq (0,0)$.

PROOF : Let σ be a collineation of π which maps $V(M(a, b))$ onto $V(\infty)$ and $V(\infty)$ onto $V(M(0, 0))$, $(a, b) \neq (0,0)$. Since C has the property that $M(x, y) \in C$ implies $-M(x, y) \in C$, a necessary condition[1] for the existence of σ is that for each $M(c, d) \in C$, $M(a, b) + M(c, d) \in C$. If $b \neq 0$ the proof is same as that of Lemma 3.1 of this note. Suppose $b \neq 0$; then the collineation $\sigma^{-1}\alpha(a)\,\sigma$ where $\alpha(a)$ is the collineation mapping $V(M(a, b))$ onto $V(M(a, b) + aI)$ fixes $V(M(0, 0))$ and moves $V(\infty)$ which is a contradiction to Lemma 3.3 of this note. Hence the Lemma is proved.

Using Lemmas 3.1, 3.2, 3.3 and 3.4 we can conclude that $V(\infty)$ and $V(M(0, 0))$ are not in the same orbit under the translation complement.

Theorem 3.5 — Every collineation of π fixes $V(\infty)$.

PROOF : The group $\langle W, H \rangle$ fixes $V(\infty)$ and is transitive on $\{V(M(a, 0)) \mid a \in GF(3^r)\}$ and $\{V(M(a, b)) \mid a, b \in GF(3^r),\ b \neq 0\}$, separately. Since $V(\infty)$ and $V(M(0, 0))$ are not in the same orbit, if any collineation ξ moves $V(\infty)$ onto a point $V(M(a, b))$, $b \neq 0$ then it moves $V(M(0, 0))$ onto a point $V(M(c, 0))$.

Let $\eta \in H$ be a collineation defined by

$$\eta : V(M(a, b)) \to V(M(a, b) + (-c)I).$$

Then $\xi\eta$ is a collineation which fixes $V(M(0, 0))$ and sends $V(\infty)$ onto $V(M(a, b))$, where $b \neq 0$. This is a contradiction to Lemma 3.3. Therefore the theorem is proved.

All the statements made in (section 4 of Narayana Rao *et al.*[1]) are valid also for $p = 3$.

Conclusion : The construction given in Narayana Rao *et al.*[1] can be extended to the prime $p = 3$ and the translation complement of any plane of order 3^{2r} of this class divides the set of distinguished points into 3 orbits of lengths, 1, 3^r, and $(3^{2r} - 3^r)$, $r \neq 1$. If $r = 1$, the plane is a Desarguesian plane of order 9.

REFERENCES

1. M. L. Narayana Rao, K. Satyanarayana and G. Vithal Rao, *Bull. Austral. Math. Soc.* **30** (1984), 56-66.
2. G. Vithal Rao, $(G, \Gamma, 2, q)$-Translation planes, Ph. D. Dissertation, Osmania University (India), 1984.