Indian J. pure appl. Math., 27(8) : 735-739, August 1996

ON NONCOOPERATIVE VECTOR EQUILIBRIUM

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(*Received* 27 *January* 1995; *after revision* 19 *February* 1996; *accepted* 11 *March* 1996)

In this paper we define a noncooperative vector equilibrium for vector-valued functions with respect to variable dominated cones in Hausdorff topological vector spaces, and prove the existence theorem for our equilibrium.

1. INTRODUCTION AND PRELIMINARIES

The existence theorem of a noncooperative (scalar) equilibrium for real-valued functions has been well-known (see e.g. Aubin[1], Fan[3] and Nash[5]). Recently, Yu[7] obtained the existence theorem of a noncooperative (scalar) equilibrium for real-valued functions in reflexive Banach spaces under coercivity conditions.

In this paper, we define a noncooperative vector equilibrium for vector-valued functions with respect to variable dominated cones in Hausdorff topological vector spaces, which is a generalization of an ordinary noncooperative (scalar) equilibrium for real-valued functions. By using the KKM-Fan theorem (see Fan[2]; also Theorem 0 below), we prove a new kind of Ky Fan inequality for vector-valued functions and establish the existence theorem for our equilibrium (see Yu[8] for dominated cones).

First, we give definitions needed for our existence theorem in section 2.

Definition 1.1 (Tanaka[6]) — Let E be a vector space and K a convex subset of E. Let Y be a Hausdorff topological vector space with a closed convex cone P such

The first and second authors were supported in part by the research institute attached to University (1993), Korea Research Foundation. The third author was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation.

that int $P \neq \phi$ and $P \neq Y$, and $g : K \to Y$ a function, where int$A$ denotes the interior of the set A.

(1) g is said to be P-convex if for any $x, y \in K$ and $\alpha \in [0, 1]$,

$$g(\alpha x + (1-\alpha)y) \in \alpha g(x) + (1-\alpha)\, g(y) - P.$$

(2) g is said to be natural quasi P-convex if for any $x, y \in K$ and $\alpha \in [0, 1]$,

$$g(\alpha x + (1-\alpha)y) \in co\{g(x),\ g(y)\}\ -\ P,$$

where coA denotes the convex hull of the set A.

Remark 1.1 : (1) Every P-convex function is natural quasi P-convex.

(2) g is natural quasi P-convex if and only if for any $x, y \in K$ and $\alpha \in [0, 1]$, there exists $\mu \in [0, 1]$ such that

$$g(\alpha x + (1-\alpha)y) \in \mu g(y) + (1-\mu)\, g(y) - P.$$

(3) When $Y = R$ and $P = R_+$ where R is the real number system and $R_+ = \{x \in R : x \geq 0\}$, the terminologies in Definition 1.1 become the usual convexity and quasiconvexity.

Definition 1.2 — Let E be a Hausdorff topological space and Y a Hausdorff topological space. Then a set-valued map $F : E \to 2^Y$ is said to be closed if the graph of F : $\{(x, y) : y \in F(x)\}$ is closed.

Now we give the definition of KKM map and the KKM-Fan theorem (Fan[2]) needed for the proof of our Lemma 2.1 in section 2.

Definition 1.3 (Fan[2]) — Let E be a vector space and K a nonempty subset of E. Then the set-valued map $G : K \to 2^E$ is called a KKM map if for each finite subset $\{x_1, ..., x_n\}$ of K, $co\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$.

Theorem 0 (KKM-Fan theorem) — Let E be a Hausdorff topological vector space, K a nonempty subset of E and $G : K \to 2^E$ a KKM map. If all the sets $G(x)$ are closed in E and if one is compact, then $\bigcap_{x \in K} G(x) \neq \phi$.

2. Noncooperative Vector Equilibrium

Let E be a Hausdorff topological vector space, K a nonempty closed convex subset of E, Y a topological vector space and $C : K \to 2^Y$ a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex cone in Y with int$C(x) \neq \phi$ and $C(x) \neq Y$.

Now by using the KKM-Fan theorem (Theorem 0 above), we prove a new kind of Ky Fan inequality for vector-valued functions, very closely related to the Ky Fan minimax inequality (Fan[4]). The proof of our inequality is a modification which is a vector version of the scalar case (Fan[4], Theorem 1).

Lemma 2.1 — Let E be a Hausdorff topological vector space, K a nonempty closed convex subset of E, Y a Hausdorff topological vector space, $G : K \times K \to Y$ a function, and $P := \bigcap_{x \in K} C(x)$ a closed convex cone such that $\mathrm{int}P \neq \phi$. Let the set-valued map $W : K \to 2^Y$ defined by $W(x) := Y \setminus (-\mathrm{int}C(x))$ be closed. Suppose that the following conditions are satisfied :

(1) for each $x \in K$, $G(x, x) \in C(x)$;

(2) for each $y \in K$, $G(\cdot\ , y)$ is natural quasi P-convex;

(3) for each $x \in K$, $G(x, \cdot)$ is continuous;

(4) there exist a nonempty compact subset B of E and $x_0 \in B \bigcap K$ such that for any $y \in K \setminus B$, $G(x_0, y) \in -\mathrm{int}C(y)$.

Then there exists $\bar{x} \in K$ such that $G(x, \bar{x}) \notin -\ \mathrm{int}\ C(\bar{x})$ for any $x \in K$.

PROOF : Define a set-valued map $V : K \to 2^K$ by for any $x \in K$,

$$V(x) = \{y \in K : G(x, y) \notin -\ \mathrm{int}C(y)\}.$$

Since $C(x) \neq Y, C(x) \bigcap (-\mathrm{int}\ C(x)) = \phi$, and so, by the condition (1), $G(x, x) \notin -\ \mathrm{int}\ C(x)$. Hence for each $x \in K$, $V(x) \neq \phi$. Moreover, V is a KKM map on K. For a contradiction, suppose that

$$y = \sum_{i=1}^{n} \alpha_i x_i,\ x_i \in K,\ \sum_{i=1}^{n} \alpha_i = 1,\ \alpha_i \geq 0 \text{ and } y \notin \bigcup_{i=1}^{n} V(x_i).$$

Then we have

$$G(x_i, y) \in -\ \mathrm{int}C(y),\ i = 1, 2, \ldots, n. \qquad \ldots (2.1)$$

Let $U = \{x \in K : G(x, y) \in -\ \mathrm{int}\ C(y)\}$, let $z_1, z_2 \in U$ and $\alpha \in [0, 1]$. Then we have

$$G(z_i, y) \in -\ \mathrm{int}\ C(y),\ i = 1, 2. \qquad \ldots (2.2)$$

By the condition (2), there exists $\mu \in [0, 1]$ such that

$$G(\alpha z_1 + (1-\alpha) z_2, y) - \mu G(z_1, y) - (1-\mu)\, G(z_2, y) \in -P. \qquad \ldots (2.3)$$

From (2.2) and (2.3), we have

$$G(\alpha z_1 + (1-\alpha)\, z_2, y) \in -P - \mathrm{int}\ C(y) - \mathrm{int}\ C(y) \subset -\mathrm{int}\ C(y).$$

Hence U is a convex subset of K, and hence, from (2.1), $y = \sum_{i=1}^{n} \alpha_i x_i \in U$. Thus we have $G(y, y) \in -\ \mathrm{int}\ C(y)$. By the condition (1), we have

$$G(y, y) \in C(y) \bigcap (-\mathrm{int}\ C(y)). \qquad \ldots (2.4)$$

Since $C(y) \neq Y$, $C(y) \bigcap (-\ \mathrm{int}\ C(y)) = \phi$. Hence (2.4) does not hold. Consequently,

V is a KKM map. By the condition (3) and the closedness of the set-valued map W, for each $x \in K$, $V(x)$ is closed. Further, by the condition (4), $V(x_0)$ is compact. Indeed, suppose that there exists $z \in V(x_0)$ such that $z \notin B$, where B is in condition (4). Since $z \in V(x_0)$, we have

$$G(x_0, z) \notin -\text{ int } C(z). \qquad \text{... (2.5)}$$

Since $z \notin B$, by the condition (4), $G(x_0, z) \notin -\text{ int}C(z)$, which contradicts (2.5). Hence $V(x_0) \subset B$. Since B is compact, $V(x_0)$ is compact. By Theorem 0 (KKM-Fan theorem), $\bigcap_{x \in K} V(x) \neq \phi$. Thus, there exists $\bar{x} \in K$ such that $G(x, \bar{x}) \notin -\text{ int } C(\bar{x})$ for any $x \in K$.

Remark 2.1 : Notice that when K is compact, the condition (4) in Lemma 2.1 is automatically satisfied with $K = B$.

Next, we define the noncooperative vector equilibrium for vector-valued functions.

In the sequel, we let $K := \prod_{i=1}^{n} K^i$ be a nonempty subset of a product space $E := \prod_{i=1}^{n} E^i$, where E^i is a Hausdorff topological vector space, Y a Hausdorff topological vector space and $G_i : K \to Y$ a function, $i = 1, ..., n$. Let for any $x = (x^1, ..., x^n) \in K$, $\hat{x^i} = (x^1, ..., x^{i-1}, x^{i+1}, ..., x^n) \in \prod_{j \neq i} K^j$ and $x = (x^i, \hat{x^i}) \in K^i \times \prod_{j \neq i} K^j$.

Definition 2.1 — Let $C : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex cone in Y with int $C(x) \neq \phi$ and $C(x) \neq Y$. Then we say that $\bar{x} \in K$ is a noncooperative vector equilibrium with respect to variable dominated cones $C(x)$ if for each $i \in \{1, ..., n\}$, we have

$$G_i(y^i, \hat{\bar{x}^i}) - G_i(\bar{x}^i, \hat{\bar{x}^i}) \notin -\text{int } C(\bar{x}) \text{ for any } y^i \in K^i.$$

Remark 2.2 : (1) When $C(x) = C$ for any $x \in K$, where C is a closed convex cone in Y with Int $C \neq \phi$ and $C \neq Y$, Definition 2.1 reduces to that of a noncooperative vector equilibrium with respect to a constant dominated cone C.

(2) When $Y = R$ and $C(x) = R_+$ for any $x \in K$, Definition 2.1 reduces to that of the ordinary noncooperative (scalar) equilibrium in (Aubin[1], Fan[3] and Nash[5]). Hence our definition is a vector of the ordinary noncooperative (scalar) equilibrium.

Now we prove the existence theorem for our noncooperative vector equilibrium with respect to variable dominated cones.

Theorem 2.1 — Let $C : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex cone in Y with int $C(x) \neq \phi$ and $C(x) \neq Y$, and $P := \bigcap_{x \in K} C(x)$ a closed convex cone with int $P \neq \phi$. Let the set-valued map W : $K \to 2^Y$ defined by $W(x) := Y \setminus (-\text{ int } C(x))$ be closed. Suppose that the following conditions are satisfied :

(1) for each $i \in \{1, ..., n\}$, K^i is closed and convex;

(2) for each $i \in \{1, ..., n\}$, the function $y^i \mapsto G_i(y^i, x^{\hat{i}})$ is P-convex;

(3) for each $i \in \{1, ..., n\}$, the function G_i is continuous; and

(4) there exist a nonempty compact subset B of E and $y_0 \in B \cap K$ such that for any $x \in K \setminus B$,

$$\sum_{i=1}^{n} [G_i(y_0^i, x^{\hat{i}}) - G_i(x^i, x^{\hat{i}})] \in - \text{ int } C(x).$$

Then there exists a noncooperative vector equilibrium with respect to variable dominated cones $C(x)$.

PROOF : Define a function $G : K \times K \to Y$ by for any $x, y \in K$,

$$G(x, y) = \sum_{i=1}^{n} [G_i(y^i, x^{\hat{i}}) - G_i(x^i, x^{\hat{i}})].$$

Then for each $x \in K$, $G(x, x) = 0 \in C(x)$. By the condition (1), K is closed and convex. By the condition (2), the function $y \mapsto G(x, y)$ is P-convex. By the condition (3), the function $x \mapsto G(x, y)$ is continuous. By the condition (4), (4) of Lemma 2.1 holds. By Lemma 2.1, there exists $\bar{x} = (\bar{x}^i, \bar{x}^{\hat{i}}) \in K$ such that for any $y \in K$,

$$G(\bar{x}, y) \notin - \text{ int } C(\bar{x}) \qquad \text{... (2.6)}$$

For each $i \in \{1, ..., n\}$ and any $y^i \in K^i$, let us take $y = (y^i, \bar{x}^{\hat{i}})$. Then from (2.6),

$$G(\bar{x}, y) = G_i(y^i, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin - \text{ int } C(\bar{x}).$$

Hence we have, for each $i \in \{1, 2, ..., n\}$,

$$G_i(y^i, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin - \text{ int } C(\bar{x}) \text{ for any } y^i \in K^i,$$

that is, $\bar{x}$ is a noncooperative vector equilibrium with respect to variable dominated cones $C(x)$.

Remark 2.3 : The above Theorem 2.1 is a vector version of the existence theorem for an ordinary noncooperative (scalar) equilibrium in (Aubin[1], Fan[3] and Nash[5]).

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