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A COMPACT GENERALIZATION OF WALSH'S TWO-CIRCLE THEOREMS FOR POLYNOMIALS AND RATIONAL FUNCTIONS

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Let $P(z)$ and $Q(z)$ be two polynomials of degree n and m respectively with prescribed zeros. In this paper we study the distribution of the zeros of the polynomial $P'(z)\,Q(z) + \beta P(z)\,Q'(z)$, where β is any given complex number and thereby present a compact generalization of Walsh's two-circle theorem for polynomials and rational functions. As an application we shall present a generalization of a mean value theorem for polynomials.

If $P(z)$ is a polynomial of degree n having all its zeros in the circle $C_1 : |z - c_1| \le r_1$ and $Q(z)$ is a polynomial of degree m having all its zeros in the circle $C_2 : |z - c_2| \le r_2$, then according to Walsh's two-circle theorem (Marden[2], p. 89; Pólya and Szegö[3], p.57) all the zeros of the polynomial

$$F(z) = P'(z)\,Q(z) + P(z)\,Q'(z)$$

lie in C_1, C_2 and a third circle

$$C : \left| z - \frac{nc_2 + mc_1}{n+m} \right| \le \frac{nr_2 + mr_1}{n+m}.$$

In the literature (for example see Marden[2] and Pólya and Szegö[3]) there exist some implicit extensions of Walsh's two-circle theorem. In this paper we shall study the distribution of the zeros of the polynomial

$$F(z) = P'(z)\,Q(z) + \beta P(z)\,Q'(z)$$

for any given complex number β and thereby present an explicit compact generalization of Walsh's two-circle theorems for polynomials and rational functions. As an application we shall also present a generalization of a mean-value theorem for polynomials. We prove :

Theorem 1 — If the locus of the zeros of a polynomial $P(z)$ of degree n is the closed interior of the circle C_1 with centre c_1 and radius r_1 and the locus of the zeros of a polynomial $Q(z)$ of degree m is the closed interior of a circle C_2 with centre c_2 and radius r_2, then for every non-zero complex number $\beta \neq -n/m$, the locus of the zeros of the polynomial

$$F(z) = P'(z)Q(z) + \beta P(z)Q'(z)$$

consists of the closed interiors of C_1 if $n > 1$, of C_2 if $m > 1$ and a third circle C with centre c and radius r where

$$c = \frac{nc_2 + \beta mc_1}{n + \beta m} \text{ and } r = \frac{nr_2 + |\beta| mr_1}{|n + \beta m|}.$$

Under these hypothesis if $\beta = -n/m$ and if the closed interiors of C_1 and C_2 have no point in common, then these two circles contain all the zeros of $F(z)$.

For the proof of Theorem 1, we need the following result, which is the coincidence theorem of Walsh[4] (see also Aziz[1]).

Lemma — Let $G(z_1, z_2, ..., z_n)$ be a symmetric n-linear form of total degree n in $z_1, z_2, ..., z_n$ and let K be a circle containing the n points $w_1, w_2, ..., w_n$. Then there exists at least one point α belonging to K such that

$$G(\alpha, \alpha, ..., \alpha) = G(w_1, w_2, ..., w_n).$$

PROOF OF THEOREM 1 : If w is any zero of $F(z)$, then

$$F(w) = P'(w)Q(w) + \beta P(w)Q'(w) = 0. \quad \text{... (1)}$$

This is an equation which is linear and symmetric in the zeros of $P(z)$ and in the zeros of $Q(z)$. By the above lemma, w will also satisfy the equation obtained by substituting into eqn. (1)

$$P(z) = (z - \alpha_1)^n \text{ and } Q(z) = (z - \alpha_2)^m$$

where α_1 is a suitably chosen point in C_1 and α_2 is a suitably chosen point in C_2. That is, w satisfies the equation

$$n(w - \alpha_1)^{n-1}(w - \alpha_2)^m + \beta m(w - \alpha_1)^n (w - \alpha_2)^{m-1} = 0.$$

Equivalently

$$(w - \alpha_1)^{n-1}(w - \alpha_2)^{m-1}((n + \beta m)w - (n\alpha_2 + \beta m\alpha_1)) = 0. \quad \text{... (2)}$$

First suppose that $\beta = -n/m$ and that the circles C_1 and C_2 have no point in common, then clearly $\alpha_1 \neq \alpha_2$ and so that $n\alpha_2 + \beta m\alpha_1 \neq 0$. Hence from (2) we get

$$w = \alpha_1, \text{ if } n > 1 \text{ or } w = \alpha_2, \text{ if } m > 1.$$

Since α_1 is a point in C_1, α_2 is a point in C_2 and w is an arbitrary zero of $F(z)$, it follows that, in this case the two cricles C_1 and C_2 contain all the zeros of $F(z)$.

Henceforth we assume that $\beta \neq -n/m$, so that $F(z)$ is a polynomial of degree $n + m - 1$. Now from (2), it follows that w has the values

$w = \alpha_1$, if $n > 1$; $w = \alpha_2$, if $m > 1$;

$$w = \frac{n\alpha_2 + \beta m\alpha_1}{n + \beta m}.$$

Clearly the first w is a point in C_1 and the second w is a point in C_2. That the third w is a point in the circle

$$C : \left| z - \frac{nc_2 + \beta mc_1}{n + \beta m} \right| \le \frac{nr_2 + |\beta| mr_1}{|n + \beta m|},$$

follows from the fact that

$$\left| \frac{n\alpha_2 + \beta m\alpha_1}{n + \beta m} - \frac{nc_2 + \beta mc_1}{n + \beta m} \right| = \left| \frac{n(\alpha_2 - c_2) + \beta m(\alpha_1 - c_1)}{n + \beta m} \right|$$
$$\le \frac{n|\alpha_2 - c_2| + |\beta| m |\alpha_1 - c_1|}{|n + \beta m|}$$
$$\le \frac{nr_2 + |\beta| mr_1}{|n + \beta m|}.$$

Since w is an arbitrary zero of $F(z)$, it follows that every zero of $F(z)$ lies in at least one of the circles C_1, C_2 and C.

Conversely we now show that any point w in or on the circles C_1, C_2 or C is a possible zero of $F(z)$. For, we may take $P(z) = (z - \alpha_1)^n$ and $Q(z) = (z - \alpha_2)^m$ and choose α_1 and α_2 as follows. If $n > 1$ and if w lies C_1, we choose $\alpha_1 = w$ and α_2 as an arbitrary point in C_2. Similarly, if $m > 1$ and w lies in C_2, we choose $\alpha_2 = w$ and α_1 as an arbitrary point in C_1. If, however, w is any point in or on C, then we may write

$$w = c + \alpha re^{i\theta}$$
$$= \frac{n}{n + \beta m} c_2 + \frac{\beta m}{n + \beta m} c_1 + \alpha e^{i\theta} \left\{ \frac{nr_2}{|n + \beta m|} + \frac{|\beta| mr_1}{|n + \beta m|} \right\},$$

$0 \le \alpha \le 1$, and associate with this w

$$\alpha_1 = c_1 + \alpha \left| \frac{\beta}{n + \beta m} \right| \frac{(n + \beta m)}{\beta} r_1 e^{i\theta}$$

$$\alpha_2 = c_2 + \alpha \frac{(n + \beta m)}{|n + \beta m|} r_2 e^{i\theta},$$

then $|\alpha_1 - c_1| \le r_1$ and $|\alpha_2 - c_2| \le r_2$, so that α_1 is a point in C_1, α_2 is a point in C_2 and

$$w = c + \alpha re^{i\theta} = \frac{n\alpha_2}{n + \beta m} + \frac{\beta m\alpha_1}{n + \beta m}.$$

This completes the proof of Theorem 1.

Remark : For $\beta = 1$, Theorem 1 reduces to Walsh's two-circles theorem for polynomials (Marden[2], p. 89). For $\beta = -1$, Theorem 1 reduces to Walsh's two-circle theorem for rational functions (Marden[2], p. 93), since in this case the zeros of $F(z) = P'(z)Q(z) - P(z)Q'(z)$ are the same as the finite zeros of the derivative of the quotient $P(z)/Q(z)$.

The following corollary follows from Theorem 1, by taking $\beta = n/m$.

Corollary — If all the zeros of a polynomial $P(z)$ of degree n lie in the circle $C_1 : |z - c_1| \le r_1$ and if all the zeros of a polynomial $Q(z)$ of degree m lie in the circle $C_2 : |z - c_2| \le r_2$, then the zeros of the polynomial

$$F(z) = mP'(z)Q(z) + n P(z) Q'(z)$$

which are not in C_1 or C_2 lie in the third circle

$$C : \left| z - \frac{c_1 + c_2}{2} \right| \le \frac{r_1 + r_2}{2}.$$

If we set in Theorem 1

$$P(z) = \prod_{j=1}^{p} (z - z_j)^{m_j}, \quad \sum_{j=1}^{p} m_j = n,$$

$$Q(z) = \prod_{j=1}^{q} (z - w_j)^{t_j}, \quad \sum_{j=1}^{q} t_j = m,$$

then

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{p} \frac{m_j}{z - z_j} \text{ and } \frac{Q'(z)}{Q(z)} = \sum_{j=1}^{q} \frac{t_j}{z - w_j}.$$

Since

$$\frac{F(z)}{P(z)\,Q(z)} = \frac{P'(z)}{P(z)} + \beta \frac{Q'(z)}{Q(z)}$$

$$= \sum_{j=1}^{p} \frac{m_j}{z - z_j} + \beta \sum_{j=1}^{q} \frac{t_j}{z - w_j} = G(z), \text{ (say)},$$

where m_j, t_j are positive integers and

$$\sum_{j=1}^{p} m_j = n, \quad \sum_{j=1}^{q} t_j = m \qquad \text{... (3)}$$

therefore, the finite zeros of $G(z)$ are the same as the zeros of $F(z)$. Thus Theorem 1 may be considered as a result concerning the finite zeros of the rational function

$$G(z) = \sum_{j=1}^{p} \frac{m_j}{z-z_j} + \beta \sum_{j=1}^{q} \frac{t_j}{z-w_j} \quad \ldots (4)$$

in which all $z_j \in C_1$, all $w_j \in C_2$, m_j, t_j are positive integers satisfying (3) and $\beta \neq -n/m$ is any given non-zero complex number.

As an application of this result, we shall deduce the following theorem which is a generalization of a mean-value theorem for polynomials.

Theorem 2 — Let the circle C_1 with centre c_1 and radius r_1 enclose all the points in which a pth degree polynomial $P(z)$ assumes the value A and let the circle C_2 with centre c_2 and radius r_2 enclose all the points in which $P(z)$ assumes the value B. Then, if n, m are arbitrary positive numbers and $\beta \neq -n/m$ is any given non-zero complex number, the circles C_1 and C_2 and a third circle C with centre $c = (nc_1 + \beta mc_2)/(n + \beta m)$ and radius $r = (nr_1 + |\beta| mr_2)/(|n + \beta m|)$ contain all the points at which $P(z)$ assumes the value $M = (nA + \beta mB)/(n + \beta m)$.

For $\beta = 1$, this reduces to a mean-value theorem for polynomials (Marden[2], p. 91).

PROOF OF THEOREM 2 : We denote by $z_1, z_2, \ldots, z_p$, the points where $P(z) = A$ and by $w_1, w_2, \ldots, w_p$, the points where $P(z) = B$. Then we have

$$P(z) - A = \prod_{j=1}^{p} (z - z_j) \text{ and } P(z) - B = \prod_{j=1}^{p} (z - w_j),$$

where $z_j \in C_1$ and $w_j \in C_2$ for all $j = 1, 2, \ldots, p$. This implies

$$\frac{P'(z)}{P(z)-A} = \sum_{j=1}^{p} \frac{1}{z-z_j} \text{ and } \frac{P'(z)}{P(z)-B} = \sum_{j=1}^{p} \frac{1}{z-w_j}. \quad \ldots (5)$$

If w denotes any point where $P(z) = M$, then

$$(n + \beta m)\, P(w) = nA + \beta mB.$$

That is

$$n(P(w) - A) + \beta m(P(w) - B) = 0.$$

Hence

$$\frac{nP'(w)}{P(w)-B} + \frac{\beta mP'(w)}{P(w)-A} = 0.$$

This gives with the help of (5)

$$\sum_{j=1}^{p} \frac{1}{w-w_j} + \frac{m\beta}{n} \sum_{j=1}^{p} \frac{1}{w-z_j} = 0.$$

Now applying the above result concerning the finite zeros of the rational function $G(z)$ defined by (4), with z_j replaced by w_j, w_j replaced by z_j, $m_j = 1 = t_j$, $q = p$ and β replaced by $m\beta/n$, it follows that w must lie in at least one of the circles C_1, C_2 and

$$\left| z - \frac{pc_1 + (m\beta/n)pc_2}{p + (m\beta/n)p} \right| \leq \frac{pr_1 + |m\beta/n| pr_2}{|p + (m\beta/n)p|} \qquad \ldots (6)$$

where $(m\beta/n) \neq -1$. That is w must lie in at least one of the circles C_1, C_2 and

$$\left| z - \frac{nc_1 + \beta mc_2}{n + \beta m} \right| \leq \frac{nr_1 + |\beta| mr_2}{|n + \beta m|}$$

which is the circle C. This completes the proof of Theorem 2.

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