Indian J. pure appl. Math., 27(8) : 791-794, August 1996

SOME INEQUALITIES FOR SELF-RECIPROCAL POLYNOMIALS

ABDUL AZIZ AND B. A. ZARGAR

Post-Graduate Department of Mathematics, University of Kashmir, Srinagar 190 006

(*Received* 27 *March* 1995; *after revision* 21 *November* 1995; *accepted* 7 *March* 1996)

Let $P(z) = P_1(z) + iP_2(z)$ be a self-reciprocal polynomial of degree n. In this paper we estimate maximum modulus of $P_1'(z) \pm P_2'(z)$ on the unit circle in terms of the maximum modulus of $P(z)$ on the unit circle and its degree and obtain a sharp result.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree n and $p'(z)$ be its derivative. Concerning the estimate of $|p'(z)|$ on the unit circle $|z| = 1$, we have

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \qquad \ldots (1)$$

Inequality (1) is an immediate consequence of *S*. Bernsteins theorem on the derivative of a trigonometric polynomial (for reference see Schaeffer[7]).

In (1) equality holds only for the polynomial $p(z) = a_n z^n$.

A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, of degree n, is said to be a 'self-reciprocal' polynomial of degree n if it satisfies

$$p(z) = z^n p(1/z), \qquad \ldots (2)$$

or equivalently,

$$a_j = a_{n-j} \text{ for } j = 0, 1, 2, \ldots, n.$$

Polynomials $p(z)$ satisfying (2) and some related functions were studied by many investigators[1-6]. Here we prove the following sharp result about polynomials satisfying (2).

Theorem 1 — If $P(z) = \sum_{j=1}^{n} (a_j + ib_j) z^j = p_1(z) + ip_2(z)$ is a polynomial of degree n satisfying (2), then

$$\max_{|z|=1} |p_1'(z) + p_2'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1} |p(z)| \qquad \dots (3)$$

and

$$\max_{|z|=1} |p_1'(z) - p_2'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1} |p(z)|. \qquad \dots (4)$$

Both the estimates are sharp with equality for the polynomial $p(z) = z^n + 2iz^{n/2} + 1$, where $n = 2m$ and m is a positive integer.

2. Lemmas

For the proof of this theorem we need the following lemmas. The first result is due to the first author (Aziz[1], Lemma 4).

Lemma 1 — Let $P(z)$ be a polynomial of degree m. If for some positive integer $n \ge m$, $P(z) = z^n \overline{P(1/\bar{z})}$, then

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|.$$

Lemma 2 — Let $P(z) = \sum_{j=0}^{n} (a_j + ib_j) z^j = P_1(z) + iP_2(z)$ be a polynomial of degree n, then for $0 \le \theta < 2\pi$,

$$|P_1(e^{i\theta})|^2 + |P_2(e^{i\theta})|^2 \le \left\{ \max_{|z|=1} |P(z)| \right\}^2. \qquad \dots (5)$$

Proof of Lemma 2 : Since

$$P_1(z) = \sum_{j=0}^{n} a_j z^j \text{ and } P_2(z) = \sum_{j=0}^{n} b_j z^j$$

are polynomials with real coefficients, therefore, we have

$$\overline{P_1(\bar{z})} = P_1(z) \text{ and } \overline{P_2(\bar{z})} = P_2(z).$$

This implies for $z = e^{i\theta}$, $0 \le \theta < 2\pi$,

$$P_1(e^{-i\theta}) = \overline{P_1(e^{i\theta})} \text{ and } P_2(e^{-i\theta}) = \overline{P_2(e^{i\theta})}. \qquad \dots (6)$$

If $M = \max_{|z|=1} |P(z)|$, then for $0 < \theta < 2\pi$, we have

$$|P_1(e^{i\theta}) + iP_2(e^{i\theta})| = |P(e^{i\theta})| \le M \quad \text{... (7)}$$

and

$$|P_1(e^{-i\theta}) + iP_2(e^{-i\theta})| = |P(e^{-i\theta})| \le M. \quad \text{... (8)}$$

Using (6) in (8) we get

$$|\overline{P_1(e^{i\theta})} + \overline{iP_2(e^{i\theta})}| \le M$$

and hence

$$\left| P_1(e^{i\theta}) - iP_2(e^{i\theta}) \right| = \left| \overline{P_1(e^{i\theta}) - iP_2(e^{i\theta})} \right|$$

$$= \left| \overline{P_1(e^{i\theta})} + \overline{iP_2(e^{i\theta})} \right| \le M. \quad \text{... (9)}$$

From (7) and (9), it follows that for $0 \le \theta < 2\pi$,

$$|P_1(e^{i\theta}) + iP_2(e^{i\theta})|^2 + |P_1(e^{i\theta}) - iP_2(e^{i\theta})|^2 \le M^2 + M^2 = 2M^2.$$

Now using the identity

$$|A + B|^2 + |A - B|^2 = 2|A|^2 + 2|B|^2,$$

we get

$$2|P_1(e^{i\theta})|^2 + 2|P_2(e^{i\theta})|^2 \le 2M^2 \text{ for } 0 \le \theta < 2\pi,$$

which is equivalent to (5) and this completes the proof of Lemma 2.

3. Proof of the Theorem

Proof of Theorem 1 — Since $P(z) \equiv P_1(z) + iP_2(z)$,

where $P_1(z)$ and $P_2(z)$ are polynomials of degree $\le n$ with real coefficients, therefore,

$$G(z) = P_1(z) + P_2(z) \text{ and } H(z) = P_1(z) - P_2(z)$$

are also polynomials of degree at most n with real coefficients. Now by hypothesis, $P(z) \equiv z^n P(1/z)$, which implies that

$$P_1(z) \equiv z^n P_1(1/z) \equiv \overline{z^n P_1(1/\overline{z})} \text{ and } P_2(z) \equiv z^n P_2(1/z) \equiv \overline{z^n P_2(1/\overline{z})},$$

and hence

$$\overline{z^n G(1/\overline{z})} \equiv G(z) \text{ and } \overline{z^n H(1/\overline{z})} \equiv H(z).$$

Applying Lemma 1 to the polynomials $G(z)$ and $H(z)$, we obtain

$$\max_{|z|=1} |P_1'(z) + P_2'(z)| = \max_{|z|=1} |G'(z)| \le \frac{n}{2} \max_{|z|=1} |G(z)| \quad \text{... (10)}$$

and

$$\operatorname*{Max}_{|z|=1} |P_1'(z) - P_2'(z)| = \operatorname*{Max}_{|z|=1} |H'(z)| \le \frac{n}{2} \operatorname*{Max}_{|z|=1} |H(z)|. \qquad \ldots (11)$$

Let $G(z)$ attain its maximum at $z = e^{i\alpha}$, $0 \le \alpha < 2\pi$, on $|z| = 1$, then with the help of Lemma 2, we get

$$\left(\operatorname*{Max}_{|z|=1} |G(z)|\right)^2 = |G(e^{i\alpha})|^2 = |P_1(e^{i\alpha}) + P_2(e^{i\alpha})|^2$$

$$\le (|P_1(e^{i\alpha})| + |P_2(e^{i\alpha})|)^2$$

$$\le 2(|P_1(e^{i\alpha})|^2 + |P_2(e^{i\alpha})|^2)$$

$$\le 2\left(\operatorname*{Max}_{|z|=1} |P(z)|\right)^2 \qquad \ldots (12)$$

and similarly we can show that

$$\left(\operatorname*{Max}_{|z|=1} |H(z)|\right)^2 \le 2\left(\operatorname*{Max}_{|z|=1} |P(z)|\right)^2. \qquad \ldots (13)$$

Using (13) in (10) and (12) in (11), we finally obtain

$$\operatorname*{Max}_{|z|=1} |P_1'(z) + P_2'(z)| \le \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|$$

and

$$\operatorname*{Max}_{|z|=1} |P_1'(z) - P_2'(z)| \le \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|,$$

which are (3) and (4) respectively. This completes the proof of the Theorem 1.

REFERENCES

1. Abdul Aziz, *Proc. Am. Math. Soc.* **89** (1983), 259-66.
2. C. Frappier and Q. I. Rahman, *Canad. J. Math.* **34** (1982), 932-44.
3. B. Datt, Ph.D. thesis, submitted to the Indian Institute of Technology, Delhi, January 1976.
4. C. Frappier, Q. I. Rahman and St. Ruscheweyh, *Trans. Am. Math. Soc.* **288** (1985), 69-99.
5. N. K. Govil, V. K. Jain and G. Labelle, *Proc. Am. Math. Soc.* **57** (1976), 238-42.
6. V. K. Jain, *J. Indian Math. Soc.* **59** (1993), 167-70.
7. A. C. Schaeffer, *Bull. Am. Math. Soc.* **47** (1941), 567-79.