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A DYNAMICAL PROBLEM OF REISSNER-SAGOCI TYPE FOR A NON-HOMOGENEOUS ELASTIC HALF-SPACE

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The problem considered is of Reissner-Sagoci type for a time-periodic rotational displacement in a circular area on the boundary of a non-homogeneous semi-infinite elastic medium whose density and shear modulus vary exponentially with depth. Using the theory of dual integral equations, the problem has been reduced to the solution of Fredholm integral equation of the second kind. Numerical computation of the surface displacement outside the circular area and of the surface shear stress inside it showing the effect of non-homogeneity has been presented for different time-periods of the applied torque.

INTRODUCTION

The mixed boundary value problem of Reissner and Sagoci[1, 2], which involves the investigation of torsional oscillations in an elastic half-space under the action of periodic shear stress applied to a circular portion of the surface of the half-space, is interesting both for its theoretical importance as well as its applicability to practical situations. Sneddon[3, 4], Gladwell[5], Noble[6] and others have formulated this problem in terms of dual integral equations. Recently Erguven[7, 8] has considered the dynamical Reissner-Sagoci problem for a radially non-homogeneous material. Pak and Saphores[9] generalised the problem and considered the effect of a buried source in the form of a torque acting on a rigid disc inside a transversely isotropic semi-infinite medium.

We consider here the dynamical Reissner-Sagoci problem for a vertically non-homogeneous medium when a time-periodic rotational displacement is prescribed on a circular region of the boundary. The problem is reduced to a pair of dual integral equations (similar to the static cases by Sneddon[3]). A restriction of frequency of the applied torque is imposed for the success of the method in non-homogeneity. The solution of the problem is finally written in terms of a Fredholm integral equation of second kind. The torque and the shear stress are calculated on the boundary of the half-space. The shear stress on the circular region is found to have the singularity in the expected form. The surface displacement outside the circular area has been plotted for different values of the non-homogeneity parameter and period of oscillation. The shear stress inside the circular region on the surface has also been plotted against distance from the origin.

Basic Equations and Boundary Conditions

Using cylindrical co-ordinates (r, θ, z) with origin at the centre of the circular region on the boundary surface of a semi-infinite elastic medium $z \geq 0$, the displacement components are given by,

$$u_r = 0, \; u_\theta = u_\theta(r, z, t), \; u_z = 0. \qquad \dots (1)$$

The two non-zero stresses are

$$\tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z}, \quad \tau_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right). \qquad \dots (2)$$

The material is assumed to be non-homogeneous, i.e. the shear modulus μ and the density ρ of the medium varying exponentially with depth, i.e.

$$\frac{\mu}{\mu_0} = \frac{\rho}{\rho_0} = e^{kz} \qquad \dots (3)$$

where k is the non-homogeneity parameter.

The only stress-equation of motion not satisfied identically is

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2}. \qquad \dots (4)$$

On using relations (2) and (3), eqn. (4) reduces to

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \left(\frac{\partial^2 u_\theta}{\partial z^2} + k \frac{\partial u_\theta}{\partial z} \right) = \frac{\rho_0}{\mu_0} \frac{\partial^2 u_\theta}{\partial t^2}. \qquad \dots (5)$$

We assume that a time-periodic rotational displacement is imparted to a circular area $r \leq r_0$ of the boundary of the half-space. We therefore consider the problem of determining the stress and displacement fields, given by eqns. (2) and (5) which satisfy the following boundary conditions :

(i) $u_\theta = \phi r e^{i\omega t}$ on $z = 0, \; r \leq r_0$ ... (6a)

where ϕ is a constant.

(ii) $\tau_{\theta z} = 0$ on $z = 0, \; r > r_0$ ... (6b)

i.e. the surface area outside $r \leq r_0$ is stress-free.

Further, the displacement u_θ obeys the regularity condition at infinity.

Derivation of Dual Integral Equations

For a time-periodic solution of eqn. (5) subject to boundary conditions (6a) and (6b) we assume that,

$$u_\theta (r, z, t) = e^{i\omega t} u(r, z). \qquad \dots (7)$$

Let us further introduce the following dimensionless quantities :

$$\bar{r} = \frac{r}{r_0}, \ \bar{z} = \frac{z}{r_0}, \ \bar{k} = k r_0, \ \bar{u} = \frac{u}{\phi r_0}, \ \bar{\omega} = \left(\frac{\rho_0}{\mu_0} \right)^{1/2} r_0 \omega . \qquad \dots (8)$$

Using (7) and (8) in eqn. (5) we obtain

$$\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}^2} + \left(\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} + \bar{k}\frac{\partial \bar{u}}{\partial \bar{z}} \right) + \bar{\omega}^2 \bar{u} = 0. \qquad \dots (9)$$

The boundary conditions (6a) and (6b) yield on using (7) and (8),

$$\bar{u} = \bar{r} \quad \text{on} \quad \bar{z} = 0, \ \bar{r} \le 1 \qquad \dots (10a)$$

$$\frac{\partial \bar{u}}{\partial \bar{z}} = 0 \ \text{on} \ \bar{z} = 0, \ \bar{r} > 1. \qquad \dots (10b)$$

Solving by the method of separation of variables the solution to eqn. (9) may be taken in the integral form as :

$$\bar{u}(\bar{r}, \bar{z}) = \int_0^\infty A(\xi)\, e^{-b\bar{z}} J_1 (\xi \bar{r})\, d\xi \qquad \dots (11)$$

where

$$b(\xi) = \bar{k}/2 + (a^2 + \xi^2)^{1/2} \qquad \dots (12a)$$

$$a^2 = \bar{k}^2/4 - \bar{\omega}^2. \qquad \dots (12b)$$

The stress $\tau_{\theta z}$ is then given by

$$\tau_{\theta z} = -\mu_0 \phi e^{i\omega t} \int_0^\infty A(\xi)\, b \exp(-(b - \bar{k})\bar{z}) J_1 (\xi \bar{r})\, d\xi. \qquad \dots (13)$$

The boundary conditions (10a) and (10b) give rise to the following dual integral equations :

$$\int_0^\infty A(\xi) J_1 (\bar{r}\xi)\, d\xi = \bar{r} \ \text{and} \ \bar{r} \le 1 \qquad \dots \textbf{(14)}$$

$$\int_0^\infty b(\xi) A(\xi) J_1 (\bar{r}\xi)\, d\xi = 0 \quad \text{for} \ \bar{r} > 1. \qquad \dots \textbf{(15)}$$

Equations (14) and (15) are equivalent to

$$\int_0^\infty \xi^{-1} B(\xi)\,(1 + M(\xi))\, J_1(\bar{r}\xi)\, d\xi = 2\bar{r}, \quad \bar{r} \le 1 \qquad \dots (16)$$

and

$$\int_0^\infty B(\xi)\, J_1(\bar{r}\xi)\, d\xi = 0, \quad \bar{r} > 1 \qquad \dots (17)$$

where

$$B(\xi) = 2b(\xi)\, A(\xi) \qquad \dots (18a)$$

$$M(\xi) = \frac{\xi}{b(\xi)} - 1. \qquad \dots (18b)$$

Clearly $M(\xi)$ satisfies the conditions :

(i) $M(0) = -1$

(ii) $M(\xi) \to 0$ as $\xi \to \infty$

(iii) $M(\xi) \sim O(\xi^{-1})$ for large ξ.

Solution of Dual Integral Equations

The dual integral equations (16) and (17) are in the standard form (Sneddon[10], p. 107) with $M(\xi)$ satisfying the necessary conditions and are therefore equivalent to a Fredholm integral equation of the second kind :

$$h_1(s) + \int_0^1 h_1(\xi)\, L(s, \xi)\, d\xi = I(s) \qquad \dots (19)$$

with

$$B(\xi) = \frac{\xi}{\sqrt{\pi}} \int_0^1 s \sin(s\xi)\, h_1(s)\, ds \qquad \dots (20)$$

where the kernel $L(s, \xi)$ is given by

$$L(s, \xi) = \frac{2\xi}{\pi s} \int_0^\infty M(y) \sin(sy) \sin(\xi y)\, dy \qquad \dots (21)$$

and $I(s)$ is given by

$$I(s) = \frac{2}{\sqrt{\pi}\, s^2} \frac{d}{ds} \int_0^s \frac{2y^3}{(s^2 - y^2)^{1/2}}\, dy = \frac{8}{\sqrt{\pi}}\, s. \qquad \dots (22)$$

Writing

$$h(s) = sh_1(s), \quad K(s, \xi) = \frac{s}{\xi} L(s, \xi) \qquad \dots (23)$$

eqns. (19), (20) and (21) become respectively,

$$h(s) + \int_0^1 K(s, \xi)\, h(\xi)\, d\xi = \frac{8s}{\sqrt{\pi}} \qquad \dots (24)$$

$$B(\xi) = \frac{\xi}{\sqrt{\pi}} \int_0^1 h(s) \sin(s\xi)\, ds \qquad \dots (25)$$

and

$$K(s, \xi) = \frac{2}{\pi} \int_0^\infty M(y) \sin(sy) \sin(\xi y)\, dy. \qquad \dots (26)$$

The stress $\tau_{\theta z}$ on the boundary surface $\bar{z} = 0$ for $\bar{r} < 1$ is therefore given by

$$\left.\frac{\tau_{\theta z}}{\phi e^{i\omega t}}\right|_{\bar{z}=0} = -\frac{\mu_0}{2} \int_0^\infty B(\xi) J_1(\bar{r}\xi)\, d\xi, \; \bar{r} > 0$$

$$= -\frac{\mu_0}{2\sqrt{\pi}} \int_0^1 h(s) \int_0^\infty \xi \sin(\xi s) J_1(\xi \bar{r})\, d\xi\, ds \qquad \dots (27)$$

(on using the value of $B(\xi)$ and reversing the order of integration)

$$= \frac{\mu_0}{2\sqrt{\pi}} \frac{d}{d\bar{r}} \int_{\bar{r}}^1 \frac{h(s)\, ds}{(s^2 - \bar{r}^2)^{1/2}}, \; \bar{r} > 0 \qquad \dots (28)$$

(Gradshteyn and Ryzhik[13], p. 730)

$\tau_{\theta z}$ therefore has a square root singularity at the edge of the circular region i.e. at $\bar{r} = 1$, as expected.

The torque T necessary to produce the prescribed rotation of the circular region $\bar{r} \le 1$, $\bar{z} = 0$ is given by

$$T = \int_0^{r_0} r\tau_{\theta z}\, 2\pi r dr$$

i.e.

$$\frac{T}{\phi e^{i\omega t}} = 2\pi r_0^3 \int_0^1 \bar{r}^2 \frac{\tau_{\theta z}}{\phi e^{i\omega t}}\, d\bar{r}$$

$$= -\mu_0 \sqrt{\pi}\, r_0^3 \int_0^1 \bar{r}^2\, d\bar{r} \int_0^1 h(s)\, ds \int_0^\infty \xi \sin(\xi s) J_1(\bar{r}\xi)\, d\xi$$

(Equation continued on page 800)

$$= -\mu_0 \sqrt{\pi}\, r_0^3 \int_0^1 h(s)\, ds \int_0^\infty \xi \sin(\xi s)\, d\xi \int_0^1 \bar{r}^2 J_1(\bar{r}\xi)\, d\bar{r}$$

$$= -\mu_0 r_0^3 \sqrt{\pi} \int_0^1 h(s)\, ds \int_0^\infty \sin(\xi s) J_2(\xi)\, d\xi$$

$$= -\mu_0 r_0^3 \sqrt{\pi} \int_0^1 h(s)\, ds \frac{\sin(2\sin^{-1} s)}{\sqrt{1-s^2}}$$

$$= -2\mu_0 r_0^3 \sqrt{\pi} \int_0^1 s h(s)\, ds. \qquad \dots (29)$$

Numerical Results and Discussion

The displacement on the surface of the half-space as given by eqns. (11) and (18a) is

$$\bar{u}(\bar{r}, 0) = \int_0^\infty \frac{B(\xi) J_1(\bar{r}\xi)\, d\xi}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}$$

where $B(\xi)$ is given by relations (24), (25) and (26). Hence,

$$\bar{u}(\bar{r}, 0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{J_1(\bar{r}\xi)}{\bar{k} + 2(\xi^2 + a^2)^{1/2}} \int_0^1 \xi h(s) \sin(\xi s)\, ds\, d\xi$$

$$= \frac{1}{\sqrt{\pi}} \int_0^1 h(s) \int_0^\infty \frac{\xi \sin(\xi s) J_1(\bar{r}\xi)}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}\, d\xi\, ds. \qquad \dots (30)$$

Again, writing,

$$2J_1(\bar{r}\xi) = H_1^{(1)}(\bar{r}\xi) + H_1^{(2)}(\bar{r}\xi)$$

where $H_1^{(1)}(\ )$ and $H_1^{(2)}(\ )$ are the Hankel functions of first and second kind of order one respectively, we have,

$$\int_0^\infty \frac{2\xi \sin(\xi s) J_1(\bar{r}\xi)}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}\, d\xi$$

$$= \int_0^\infty \frac{\xi \sin(\xi s) H_1^{(1)}(\bar{r}\xi)}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}\, d\xi + \int_0^\infty \frac{\xi \sin(\xi s) H_1^{(2)}(\bar{r}\xi)}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}\, d\xi$$

$$= I_1 + I_2.$$

The evaluation of integrals I_1 and I_2 are discussed in Appendix A.1.

We finally have

$$\int_0^\infty \frac{\xi J_1(\bar{r}\xi)\sin(\xi s)\,d\xi}{\bar{k}+2(\xi^2+a^2)^{1/2}} = \frac{4}{\pi}\int_a^\infty \frac{y(y^2-a^2)^{1/2}K_1(y\bar{r})\sinh(ys)}{\bar{k}^2+4(y^2-a^2)}\,dy$$

for $\bar{r} > s$ ... (31)

where K_1 ( ) is the modified Bessel function of second kind, so that

$$\bar{u}(\bar{r},0) = \frac{4}{(\pi)^{3/2}}\int_0^1 h(s)\int_a^\infty \frac{y(y^2-a^2)^{1/2}K_1(y\bar{r})\sinh(ys)}{\bar{k}^2+4(y^2-a^2)}\,dy\,ds. \quad \text{... (32)}$$

Numerical values of $h(s)$ have been calculated by reducing the integral equation (24) to a system of algebraic equations, where the kernel $K(\alpha, \beta)$, given by equation (26) has been evaluated by the method outlined in Appendix A. 2. The dimensionless displacement $\bar{u}$ ($\bar{r}$, 0) is evaluated from eqn. (32) for different values of the non-homogeneity parameter $\bar{k}$ at different distances $\bar{r}$ from the origin, with $\bar{\omega} = \left(\frac{\bar{k}^2}{4} - a^2\right)^{1/2}$ suitably small. Figures 1-3 display the numerical values in the cases for $\bar{r} > 1.0$,

(i) $\bar{\omega}^2 = 0.1$, $\bar{k} = 1.0, 2.0$

(ii) $\bar{\omega}^2 = 0.2$, $\bar{k} = 1.0, 2.0$

(iii) $\bar{\omega}^2 = 0.1$, $\bar{k} = 1.0, 1.5, 2.0$

We notice from Figs. 1-3 that the effect of the non-homogeneity on the displacement is to decrease the amplitude, which becomes more pronounced as the distance $\bar{r}$ from the origin increases.

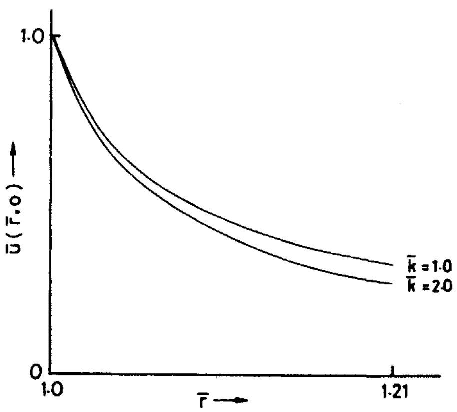

FIG. 1. Variation of $\bar{u}(\bar{r}, 0)$ with $\bar{r}$ (> 1.0) for $\bar{\omega}^2 = 0.1$.

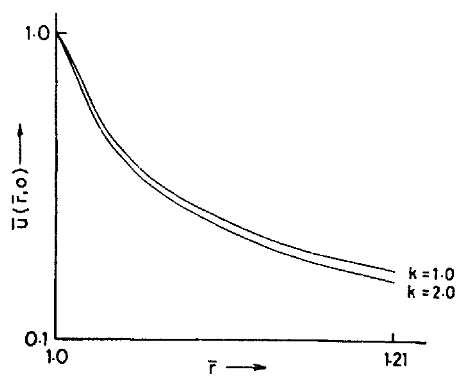

FIG. 2. Variation of $\bar{u}(\bar{r}, 0)$ with $\bar{r}$ for $\bar{\omega}^2 = 0.2$.

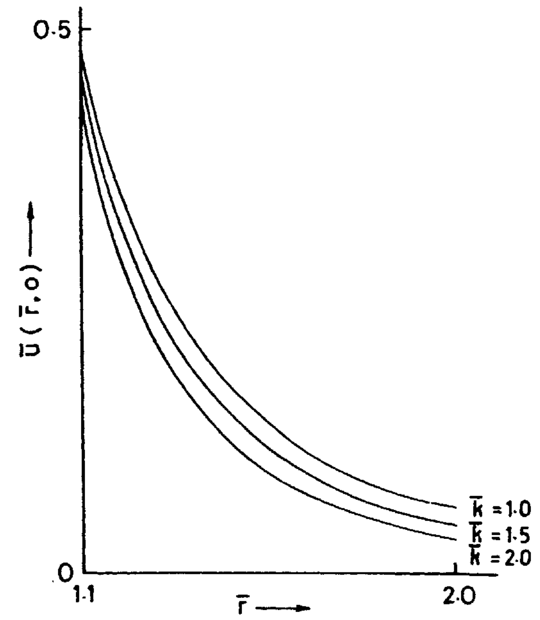

FIG. 3. Variation of $\bar{u}(\bar{r}, 0)$ with $\bar{r}$ (> 1.1) for different $\bar{k}$, with $\bar{\omega}^2 = 0.1$.

Further, from eqn. (28) we evaluate the shear stress $\tau_{\theta z}$ on the surface $z = 0$, for different $\bar{k}$ and $\bar{\omega}$. The variation of $\tau_{\theta z}/\phi\, \mu_0\, e^{i\omega t}$ on $z = 0$ with $\bar{r}$ (< 1.0) is shown in Fig. 4, for the cases :

$$\bar{k} = 1.0,\ 2.0,\ \bar{\omega}^2 = 0.1,\ 0.2.$$

The curves in the figure agree qualitatively with those of Sagoci[2] for small frequency.

It may be mentioned that the above solution may be interpreted as the approximate solution of the finite slab case provided the thickness of the slab is sufficiently large compared to the radius of the circular area over which torsion is applied.

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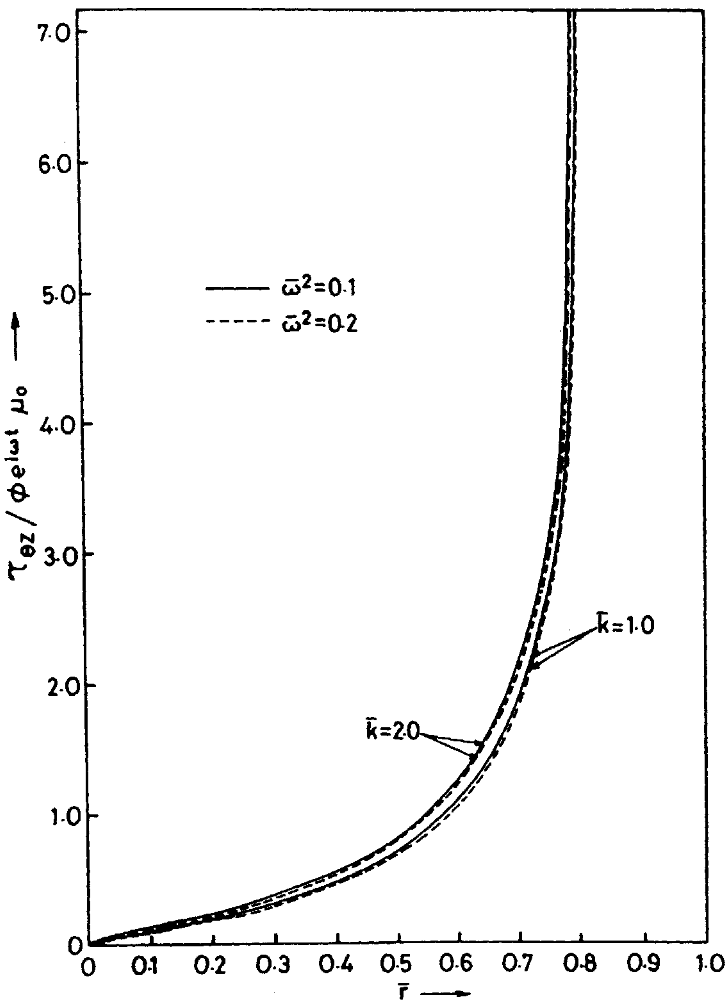

FIG. 4. Variation of $\tau_{\theta z}/\phi\,\mu_0 e^{i\omega t}$ with $\bar{r}$ (< 1.0).

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APPENDIX A

A.1. *Evaluation of* I_1 and I_2

We have

$$I_1 = \text{Im} \int_0^\infty \frac{\xi H_1^{(1)}(\xi \bar{r})\, e^{i\xi s}\, d\xi}{\bar{k} + 2(\xi^2 + a^2)^{1/2}}$$

and
$$I_2 = \text{Im} \int_0^\infty \frac{\xi H_1^{(2)}(\xi \bar{r})\, e^{i\xi s}\, d\xi}{\bar{k} + 2(\xi^2 + a^2)^{1/2}} .$$

To evaluate I_1 we take a contour Γ_1 as shown in the upper half of Fig. 5, avoiding the branch point ia. Since $H_1^{(1)}(\xi \bar{r}) \sim O(\xi^{-1/2})$ for ξ, we have on using Jordan's lemma,

$$\int_{c_1} \frac{\xi H_1^{(1)}(\xi \bar{r})\, e^{i\xi s}\, d\xi}{\bar{k} + 2(\xi^2 + a^2)^{1/2}} \to 0 \text{ as } R \to \infty, \text{ since } s + \bar{r} > 0.$$

Therefore,

$$\text{Im}\left[- \int_0^a \frac{iyH_1^{(1)}(iy\bar{r})\, e^{-ys}\, i}{\bar{k} + 2(a^2 - y^2)^{1/2}} dy - \int_a^\infty \frac{iyH_1^{(1)}(iy\bar{r})\, e^{-ys}\, i}{\bar{k} + 2i\,(y^2 - a^2)^{1/2}} dy + \int_0^\infty \frac{xH_1^{(1)}(x\bar{r})\, e^{ixs}}{\bar{k} + 2(x^2 + a^2)^{1/2}} dx \right] = 0$$

i.e.
$$\text{Im}\left[\int_0^a \frac{yH_1^{(1)}(iy\bar{r})\, e^{-ys}}{\bar{k} + 2(a^2 - y^2)^{1/2}} dy + \int_a^\infty \frac{yH_1^{(1)}(iy\bar{r})\, e^{-y\bar{s}}\,(\bar{k} - 2i(y^2 - a^2)^{1/2})}{\bar{k}^2 + 4\,(y^2 - a^2)} dy + \int_0^\infty \frac{xH_1^{(1)}(x\bar{r})\, e^{ixs}}{\bar{k} + 2(x^2 + a^2)^{1/2}} dx \right] = 0.$$

For evaluation of I_2, a contour Γ_2 as shown in Fig. 5 is used, avoiding the branch point $-ia$. Proceeding as before, we obtain

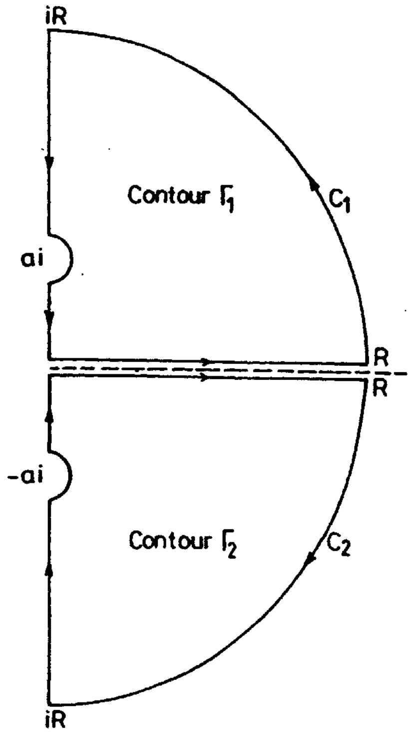

FIG. 5. Complex ξ-plane.

$$\mathrm{Im}\left[\int_a^\infty \frac{yH_1^{(2)}(-iy\bar{r})\,e^{ys}}{\bar{k}^2+4(y^2-a^2)}\,(\bar{k}+2i(y^2-a^2))^{1/2}\,dy + \int_0^a \frac{yH_1^{(2)}(-iy\bar{r})\,e^{ys}}{\bar{k}+2(a^2-y^2)^{1/2}}\,dy + \int_0^\infty \frac{xH_1^{(2)}(x\bar{r})\,e^{ixs}}{\bar{k}+2(x^2+a^2)^{1/2}}\,dx\right] = 0.$$

On using following formulae[14],

$$H_1^{(1)}(iy\bar{r}) = -\frac{2}{\pi}K_1(y\bar{r}),\quad H_1^{(2)}(-iy\bar{r}) = -\frac{2}{\pi}K_1(y\bar{r})$$

$$I_1+I_2 = \frac{8}{\pi}\int_a^\infty \frac{y(y^2-a^2)^{1/2}K_1(y\bar{r})\sinh(ys)}{\bar{k}^2+4(y^2-a^2)}\,dy.$$

A.2. *Evaluation of* $K(\alpha,\beta)$

The kernel $K(\alpha,\beta)$ in eqn. (24) may be expressed as

$$\frac{\pi}{2}K(\alpha,\beta) = \int_0^\infty M(\xi)\sin\alpha\xi\,\sin\beta\xi\,d\xi$$

$$= \frac{1}{2}\,\text{Real}\left[\int_0^\infty (M(\xi)\, e^{i(\alpha-\beta)\xi} - M(\xi)\, e^{i(\alpha+\beta)\xi})\, d\xi\right]$$

where $$M(\xi) = \frac{\xi - \frac{\bar{k}}{2} - (\xi^2 + a^2)^{1/2}}{\frac{\bar{k}}{2} + (\xi^2 + a^2)^{1/2}}.$$

For $\alpha > \beta$,

$$\frac{\pi}{2} K(\alpha, \beta) = \frac{1}{2}\,\text{Real}\left[\int_0^\infty M(\xi)\, e^{i(\alpha-\beta)\xi}\, d\xi - \int_0^\infty M(\xi)\, e^{i(\alpha+\beta)\xi}\, d\xi\right].$$

Taking a contour as Γ_1 in Fig. 5, avoiding the branch point ia, we evaluate above integrals. Since $M(\xi) = O(\xi^{-1})$ for large ξ, using Jordan's lemma,

$$\int_{c_1} M(\xi)\, e^{-(\alpha-\beta)\xi}\, d\xi \to 0 \text{ as } R \to \infty.$$

Therefore for $\alpha > \beta$,

$$\frac{\pi}{2} K(\alpha, \beta) = \frac{1}{2}\,\text{Real}\left[\int_0^a M(iy)\, e^{-(\alpha-\beta)y}\, i dy + \int_a^\infty M(iy)\, e^{-(\alpha-\beta)y}\, i dy - \int_0^a M(iy)\, e^{-(\alpha+\beta)y}\, i dy - \int_0^\infty M(iy)\, e^{-(\alpha+\beta)y}\, i dy\right]$$

$$= \frac{1}{2}\left[\int_0^a \frac{-y e^{-(\alpha-\beta)y}}{\frac{\bar{k}}{2} + (a^2 - y^2)^{1/2}} + \int_a^\infty \frac{-\frac{\bar{k}}{2} y e^{-(\alpha-\beta)y}}{\frac{\bar{k}^2}{4} + (y^2 - a^2)}\, dy \right.$$

$$\left. - \int_0^a \frac{-y e^{-(\alpha+\beta)y}}{\frac{\bar{k}}{2} + (a^2 - y^2)^{1/2}}\, dy - \int_a^\infty \frac{\frac{\bar{k}}{2} y e^{-(\alpha+\beta)y}}{\frac{\bar{k}^2}{4} + (y^2 - a^2)}\, dy\right]$$

$$= -\int_0^a \frac{y e^{-\alpha y} \sinh(\beta y)\, dy}{\frac{\bar{k}}{2} + (a^2 - y^2)^{1/2}} - \frac{\bar{k}}{2}\int_0^\infty \frac{y e^{-\alpha y} \sinh(\beta y)\, dy}{\frac{\bar{k}^2}{4} + (y^2 - a^2)}.$$

For $\alpha < \beta$, it is similarly shown that

$$\frac{\pi}{2} K(\alpha, \beta) = -\int_0^a \frac{y e^{-\beta y} \sinh(\alpha y)\, dy}{\frac{\bar{k}}{2} + (a^2 - y^2)^{1/2}} - \frac{\bar{k}}{2}\int_a^\infty \frac{y e^{-\beta y} \sinh(\alpha y)\, dy}{\frac{\bar{k}^2}{4} + (y^2 - a^2)}.$$