Indian J. pure appl. Math., 27(9) : 843-851, September 1996

ASYMPTOTIC BEHAVIOUR OF IMPULSIVE DELAY DIFFERENTIAL EQUATION

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(*Received* 24 *October* 1995; *after revision* 17 *January* 1996; *accepted* 26 *March* 1996)

Consider the delay differential equation

$$\begin{cases} \dot{x}(t) + \sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) = 0, & t \neq t_j,\ t \geq t_0, \\ x(t_j^+) - x(t_j) = b_j\, x(t_j), & j = 1, 2, \ldots, \end{cases} \qquad \ldots (A)$$

where $p_i \in C([t_0, \infty), \mathbf{R})$, $\tau_i \geq 0$ for $i = 1, 2, \ldots, n$ and $b_j \in (-\infty, \infty)$ for $j = 1, 2, \ldots$. When $p_i(t) \geq 0$ for t sufficiently large, this paper gives sufficient conditions ensuring that every solution of (A) tends to zero as $t \to \infty$, and some criteria of asymptotic behaviour of the nonoscillatory solutions of (A) with oscillating coefficients, which improve and develop some of the known results obtained in the literature[1, 6-10]. Finally, we discuss the existence of nonoscillatory solutions.

1. INTRODUCTION

In this paper, we consider the asymptotic behaviour of impulsive delay differential problem

$$\begin{cases} \dot{x}(t) + \sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) = 0, & t \neq t_j,\ t \geq t_0, \\ x(t_j^+) - x(t_j) = b_j\, x(t_j), & j = 1, 2, \ldots; \end{cases} \qquad \ldots (1)$$

where $p_i \in C([t_0, \infty), \mathbf{R})$, $\tau_i \geq 0$ for $i = 1, 2, \ldots, n$ and $b_j \in (-\infty, \infty)$ are constants for $j = 1, 2, \ldots$. Let $\tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$ and $t_0 < t_1 < t_2 < \ldots < t_j \to \infty$.

A nontrivial solution of (1) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise it will be called oscillatory.

Shen and Wang[10] discussed the following problem :

$$\begin{cases} \dot{x}(t) + p(t)\,x(t-\tau) = 0, & t \neq t_j,\ t \geq 0, \\ x(t_j^+) - x(t_j) = I_j(x(t_j)), & j = 1, 2, \ldots, \end{cases} \qquad \ldots (2)$$

and obtained

Theorem A — Assume that $p(t) \in C(\mathbf{R}_+, \mathbf{R}_+)$, $|I_k(u)| \leq b_k |u|$, $k = 1, 2, \ldots$ and

$$\int_0^\infty p(s)\,ds = \infty \text{ and } \sum_{k=1}^\infty b_k < \infty.$$

Then all nonoscillatory solutions of (2) tend asymptotically to zero.

In section 2, utilizing the superior limit and inferior limit of solutions of eqn. (1), which is a new method different from one used in Theorem A, we assert that for a general case, eqn. (1) possesses the same conclusion as Theorem A. Furthermore, we introduce a comparison result that guarantees that every solutions of (1) tends asymptotically to zero under the following conditions for sufficiently large t :

$$p_i(t) \geq 0 \ (i = 1, 2, \ldots, n) \text{ and } \sum_{i=1}^n p_i(t) > 0. \qquad \ldots (3)$$

Theorem 1 — Assume that (3) is satisfied and

$$\int_{t_0}^\infty \sum_{i=1}^n p_i(s)\,ds = \infty, \quad \sum_{k=1}^\infty b_k^* < \infty, \qquad \ldots (4)$$

where $b_k^* = \max\{b_k, 0\}$, $k = 1, 2, \ldots$. Moreover, assume that eqn. (1) has a nonoscillatory solution. Then every solution of (1) tends to zero as $t \to \infty$.

Up to now, there are only some papers studying the impulsive DDE (for this purpose see papers by Byszewski[3-5], and by Bainov and Minchev[2], still less one with oscillatory coefficients. Section 3 is concerned with the case of eqn. (1) with oscillating coefficients, and with

$$p_i(t) = p_i^+(t) - p_i^-(t),$$

where $p_i^+(t) = \max\{p_i(t), 0\}$ and $p_i^-(t) = \{\max\{-p_i(t), 0\}$, $i = 1, 2, \ldots, n$. Then the following result holds :

Theorem 2 — Assume that

$$\sum_{i=1}^n p_i^-(t) \in L^1[t_0, \infty), \ b_j \geq 0, \ \sum_{i=1}^\infty b_j < \infty \qquad \ldots (5)$$

and

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i^+(s)\, ds = \infty. \qquad \text{... (6)}$$

Then every nonoscillatory solution of (1) tends to zero as $t \to \infty$.

In addition, section 4 discuss the existence of nonoscillatory solutions.

2. The Proof of Theorem 1

First, we shall show a theorem which improves and generalizes Theorem A.

Theorem A′ — If (4) holds and $p_i \in C([t_0, \infty), \mathbf{R}^+)$, $i = 1, 2, ..., n$, then every nonoscillatory solution of eqn. (1) tends to zero as $t \to \infty$.

Proof : From (4), we can define a subset $\{t_k^*\}$ of $\{t_j\}$ corresponding to $\{b_k^*\}$. Assume that eqn. (1) has an eventually positive solution $z(t)$ such that for sufficiently large T,

$$z(t) > 0 \text{ and } \dot{z}(t) \le 0, \; t \ne t_j \quad t \ge T, \qquad \text{... (7)}$$

that is $z(t)$ is decreasing in $(t_j, t_{j+1}]$ for $t_j \ge T$, $j = m, m+1, ...,$ with $m \in \mathrm{N}$. It is easy to see that $z(t)$ is also decreasing in $(t_k^*, t_{k+1}^*]$ for $t_k^* \ge T$, $k \ge m$. Hence, for $t \ge t_k^*$

$$\begin{aligned} z(t) &\le z(t_k^{*+}) \le (1 + b_k^*)\, z(t_k^*) \le (1 + b_k^*)\, z(t_{k-1}^{*+}) \\ &\le (1 + b_k^*)(1 + b_{k-1}^*)\, z(t_{k-1}^*) \qquad \text{... (8)} \\ &\le (1 + b_k^*)(1 + b_{k-1}^*) \,....\, (1 + b_m^*)\, z(t_m^*). \end{aligned}$$

In view of (4), $0 < \Pi\, (1 + b_k^*) < \infty$. Thus, from (8), there exists a constant $M > 0$ such that

$$z(t) < M \text{ for } t \ge T.$$

Now, we claim that $\liminf_{t \to \infty} z(t) = 0$. Otherwise, set

$$\liminf_{t \to \infty} z(t) = l > 0. \qquad \text{... (9)}$$

Then there exists $T_1 \ge T$ such that $z(t) \ge l/2$ for $t - \tau > T_1$. So

$$\begin{aligned} 0 &= \dot{z}(t) + \sum_{i=1}^{n} p_i(t)\, z(t - \tau_i) \\ &\ge \dot{z}(t) + \frac{l}{2} \sum_{i=1}^{n} p_i(t). \end{aligned}$$

Integrating from t to ∞ with $t \ge T_1$, we get

$$l - M \sum_{k \geq m} b_k^* - z(t) + \frac{l}{2} \int_t^{\infty} \sum_{i=1}^{n} p_i(s)\, ds \leq 0$$

which, in view of (4), implies a contradiction that completes our claim.

Next, we have to prove that $\limsup_{t \to \infty} z(t) = 0$. From (7), we can choose a subsequence $\{\xi_k\}_1^{\infty}$ of $\{t_k^*\}_m^{\infty}$ such that

$$\lim_{k \to \infty} z(\xi_k) = 0, \qquad \dots (10)$$

and similarly, we can find another subsequence $\{\eta_k^+\}_1^{\infty}$ of $\{t_k^{*+}\}_m^{\infty}$ between ξ_k and ξ_{k+1}, $k = 1, 2, \dots$, such that $\lim_{k \to \infty} z(\eta_k^+) = \limsup_{t \to \infty} z(t)$. Assume that $b_{\underline{k}}^*, b_{\overline{k}}^*$ correspond to the moments ξ_k, η_k of impulsive effect, respectively. Then, from (1) and (7), it follows that

$$0 < z(\eta_{\overline{k}}^+) \leq (1 + b_{\overline{k}}^*)\, z(\eta_k) \leq (1 + b_{\overline{k}}^*)\, z(\eta_{k-1}^+)$$

$$\leq (1 + b_{\overline{k}}^*)(1 + b_{\overline{k}-1}^*) \dots (1 + b_{\underline{k}}^*)\, z(\xi_k), \; k = 1, 2, \dots,$$

and (10) yield $\lim_{k \to \infty} z(\eta_k^+) = 0$. Therefore, $z(t) \to 0$ as $t \to \infty$. The proof of Theorem A′ is completed.

Remark 1 : If there are only finite $b_j > 0$ in eqn. (1), the asymptotic behaviour of (1) can be treated as the delay differential equation without impulse in Theorem A′.

The following Lemmas are needed for the proof of Theorem 1 :

Lemma 1 — Assume that $z(t)$ is a nonoscillatory solution of eqn. (1). Set

$$w(t) = x(t)/z(t), \quad t \geq T,$$

where $x(t)$ is any solution of (1) and $T \geq t_0$ is such that $z(t) \neq 0$ for $t \geq T$. Then

$$\dot{w}(t) = \sum_{i=1}^{n} p_i(t) \frac{z(t-\tau_i)}{z(t)} (w(t) - w(t-\tau_i)), \quad t \geq T, \quad t \neq t_j.$$

Remark 2 : It follows from (1) that for $t = t_j$, $j = 1, 2, \dots$

$$w(t_j^+) = \frac{x(t_j^+)}{z(t_j^+)} = \frac{x(t_j) + b_j\, x(t_j)}{z(t_j) + b_j\, z(t_j)} = \frac{x(t_j)}{z(t_j)} = w(t_j).$$

Consequently w is continuous for $t \geq T$.

Lemma 2 — Assume that (3) is satisfied. Let $z(t)$ be a nonoscillatory solution of eqn. (1) and let $x(t)$ be any oscillatory solutions. Then there exists $K > 0$ such that eventually

$$|x(t)| \leq K|z(t)|.$$

PROOF : Without loss of generality, suppose that $z(t) > 0$ for $t \geq T$. Hence, using the function w introduced in Lemma 1, we have to prove that w is bounded. Otherwise, in view of Remark 1, w is an oscillatory and continuous function, and consequently there exists $t^* \geq T + \tau$ such that either

$$\dot{w}(t) \leq 0 \text{ and } w(t^*) > w(s) \text{ for } T \leq s < t^*,$$

or

$$\dot{w}(t^*) \geq 0 \text{ and } w(t^*) < w(s) \text{ for } T \leq s < t^*,$$

in which t^* may be an element of $\{t_j^+\}$. By Lemma 1, there exists a contradiction with (2).

Lemma 2 means that the nonoscillatory solutions of (1) dominate the growth of its oscillatory solutions. Hence, combining Theorem A′ with condition (3), it is easy to see that Theorem 1 holds.

3. The Proof of Theorem 2

Consider eqn. (1) with oscillating coefficients. Let us show the following lemma :

Lemma 3 — Assume that (5) is satisfied. Then every nonoscillatory solution of (1) tends to some constant as $t \to \infty$.

PROOF : Without loss of generality, suppose that $z(t)$ is an eventually positive solution of (1). There exists $T \geq t_0$ such that

$$z(t - \tau) > 0 \text{ for } t \geq T. \qquad \dots (11)$$

In view of (5), we may assume that for $T \geq T_0$, $t_j \geq T$ with $j \geq m$,

$$\int_T^\infty \sum_{i=1}^n p_i^-(s)\,ds = \alpha \text{ and } \sum_{j=m}^\infty b_j = \beta$$

where $\alpha + \beta < 1$. Then we claim that $z(t)$ is bounded above. Otherwise, there exists a sequence $\{\xi_k\}$ with $\xi_k \geq T$ such that

$$\lim_{k\to\infty} \xi_k = \infty,\ \lim_{k\to\infty} z(\xi_k) = \infty,\ z(\xi_k) = \max_{t \leq \xi_k} z(t), \qquad \dots (12)$$

where ξ_k may be the element of $\{t_j^+\}$. Equations (1) and (11) imply the inequality

$$\dot{z}(t) \leq \sum_{i=1}^n p_i^-(t)\, z(t - \tau_i), \quad t \geq T.$$

Noting (12) and integrating from T to ξ_k, we obtain

$$z(\xi_k) - z(\xi_k) \sum_{j=m}^{\infty} b_j - z(t) \le \alpha z(\xi_k).$$

Thus,

$$z(\xi_k) \le \frac{z(T)}{1 - \alpha - \beta}$$

which is in contradiction with (12). Consequently our claim is completed.

Furthermore, in view of (5), we have

$$\sum_{i=1}^{n} p_i^- (t)\, z(t - \tau_i) \in L^1 [T, \infty). \qquad \text{... (13)}$$

Integration (1) from T to t with $t > T$ yields

$$\int_{T'}^{t} \sum_{i=1}^{n} p_i^+ (s)\, z(s - \tau_i)\, ds \le \int_{T}^{\infty} \sum_{i=1}^{n} p_i^- (s)\, z(s - \tau_i)\, ds + \sup_{s \ge T} z(s)\,(2 + \beta). \qquad \text{... (14)}$$

So, $\sum_{i=1}^{n} p_i^+ (t)\, z(t - \tau_i) \in L^1 [T, \infty)$. Then $\dot{z} \in L^1 [T, \infty)$. Hence, integrating (1) from T to t with $T < t_{m^*} < t < t_{m^*+1}$, we get

$$z(t) = \sum_{j=m}^{m^*} b_j\, z(t_j) + z(T) - \int_{T}^{t} \sum_{i=1}^{n} p_i(s)\, z(s - \tau_i) ds.$$

Since $\sum_{j=1}^{\infty} b_j$ is convergent and $z(t_j)$ is bounded, $\sum_{j=m}^{\infty} b_j\, z(t_j)$ is also convergent. Moreover, from (13) and (14), it follows that $\lim_{t \to \infty} z(t)$ exists. The proof of Lemma 3 is completed.

Proof of Theorem 2

Assume, for the sake of contradiction, that eqn. (1) has an eventually positive solution $z(t)$ which does not tends to zero as $t \to \infty$. In view of Lemma 3, set

$$\lim_{t \to \infty} z(t) = l > 0.$$

Thus, for sufficiently large t with $t + \tau \ge T$, we have $l/2 \le z(t) \le 2l$. Then

$$0 = \dot{z}(t) + \sum_{i=1}^{n} p_i(t)\, z(t - \tau_i)$$

$$\geq \dot{z}(t) + \frac{l}{2} \sum_{i=1}^{n} p_i^+(t) - 2l \sum_{i=1}^{n} p_i^-(t).$$

Integration from T to ∞ yields

$$l - 2l \sum_{j=m}^{\infty} b_j - z(T) + \frac{l}{2} \int_T^{\infty} \sum_{i=1}^{n} p_i^+(s)\, ds - 2l \int_T^{\infty} \sum_{i=1}^{n} p_i^-(s)\, ds \leq 0.$$

From (5) and (6), we get a contradiction. Therefore, the proof of Theorem 2 is completed.

4. Existence of Nonoscillatory Solutions

Consider the following equation without impulse corresponding to eqn. (1),

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) = 0, \; t \geq t_0, \qquad \dots (15)$$

where $p_i \geq 0$, $i = 1, 2, .., n$. Zhao and Zhang[12] obtained a comparison result.

Lemma 4 — Assume that (3) is satisfied and $b_j \geq 0$, $j = 1, 2, \dots$. Then eqn. (15) has a nonoscillatory solution implies that eqn. (1) has also one.

So, our main aim in this section is to establish a necessary and sufficient result for the positive solutions of eqn. (15). Hence, combining with Lemma 4, we easily get the existence of nonoscillatory solutions of eqn. (1).

Theorem 3 — Equation (15) has a positve solution if and only if there exists $k_i(t) \in C([t_0, \infty), \mathbf{R}_+)$, $i = 1, 2, \dots, n$ such that

$$\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s)\, k_i(s)\, ds \leq \ln k_j(t), \quad t \geq t_0, \; j = 1, 2, \dots, n. \qquad \dots (16)$$

Proof : *Sufficiency* — Let $h_i(t) = \min\{t_0, t-\tau_i\}$, $H_i(t) = \max\{t_0, t-\tau_i\}$, and take an initial condition such that $x(t) \geq 0$, $t \in [t_0 - \tau, t_0]$ and $x(t_0) > 0$, $x(s) \leq x(t_0)$, $s \in [t_0 - \tau, t_0]$. Set

$$\beta(t) = -\sum_{i=1}^{n} p_i(t)\, k_i(t), \; \gamma(t) = 0, \text{ for } t \geq t_0.$$

For all $\delta \in C([t_0, \infty), \mathbf{R})$ such that $\beta(t) \leq \delta(t) \leq \gamma(t)$, $t \geq t_0$, it follows from (16) that

$$-\int_{H_j(t)}^{t} \delta(s)\, ds \leq \int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s)\, k_i(s)\, ds \leq \ln k_j(t),$$

where $t \geq t_0$, $j = 1, 2, \dots, n$. Hence,

$$\gamma(t) = 0 \geq -\sum_{i=1}^{n} p_i(t) \frac{x(h_i(t))}{x(t_0)} \exp\left(-\int_{H_i(t)}^{t} \delta(s)\, ds\right)$$

$$\geq -\sum_{i=1}^{n} p_i(t)\, k_i(t) = \beta(t).$$

In view of Theorem 3.1.1 in Györi and Ladas[11], the solution satisfying the initial condition is a positive solution of eqn. (15).

Necessity — Assume that $x(t)$ is a positive solution of eqn. (15). Let

$$\alpha(t) = \frac{\dot{x}(t)}{x(t)}, \quad t \geq t_0.$$

So,

$$x(t) = x(t_0) \exp\left(\int_{t_0}^{t} \alpha(s)\, ds\right), \quad t \geq t_0.$$

Hence, it follows from Theorem 3.1.1 in Györi and Ladas[11] that

$$\alpha(t) + \sum_{i=1}^{n} p_i(t) \frac{x(h_i(t))}{x(t_0)} \exp\left(-\int_{H_i(t)}^{t} \alpha(s)\, ds\right) = 0. \quad \text{... (17)}$$

Set

$$k_i(t) = \exp\left(-\int_{H_i(t)}^{t} \alpha(s)\, ds\right).$$

Integrating (17) from $H_i(t)$ to t, we get

$$\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{x(h_i(s))}{x(t_0)} k_i(s)\, ds$$

$$= \int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s)\, k_i(s)\, ds = -\int_{H_j(t)}^{t} \alpha(s)\, ds$$

$$= \ln k_j(t), \quad t \geq t_0,\ j = 1, 2, \ldots, n$$

which complete the proof of Theorem 3.

Theorem 3 implies a well-known result that if $k_i(t) \equiv e$, formula (16) changes to

$$\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds \leq \frac{1}{e}, \quad t \geq t_0,$$

which is a sufficient condition ensuring that eqn. (15) has positive solutions.

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