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THE SEIFERT-VAN KAMPEN THEOREM FOR THE GROUP OF GLOBAL SECTIONS

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Let X be the union of the subspaces U_1 and U_2 that are both open, path connected, $U_{12} = U_1 \bigcap U_2 \neq \phi$ and U_{12} is also path connected. In this paper, we first construct the sheaf H of the fundamental groups of a path connected space and give the characteristic features of H. Then, the homomorphisms and global sections of the sheaf H are explored. Finally, it is proved that if the groups of global sections $\Gamma(U_{12}, H_{12}) = \langle S; R \rangle$, $\Gamma(U_1, H_1) = \langle S_1; R_1 \rangle$ and $\Gamma(U_2, H_2) = \langle S_2; R_2 \rangle$ are given, then the group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_1 \bigcup S_2$ and the relations $R_1 \bigcup R_2 \bigcup R_S$. As a result of this, the sheaf H, especially the fundamental group $\pi_1(X, x)$ was easily calculated for any $x \in X$.

1. INTRODUCTION

Let X be a path connected space and H_x be the fundamental group of X based for any $x \in X$, that is $H_x = \pi_1(X, x)$. Let $X = (X, c)$ be a pointed topological space, for an arbitrary fixed point $c \in X$. Let us denote the disjoint union of all fundamental groups obtained for each $x \in X$, by H, i.e., $H = \bigvee_{x \in X} H_x$. H is a set over X and the mapping $\varphi : H \rightarrow X$ defined by $\varphi(\sigma_x) = \varphi([\alpha]_x) = x$, for any $\sigma_x = [\alpha]_x \in H_x \subset H$, is onto.

We introduce a topology on H as follows :

Let $W \subset X$ be an open set. Define a mapping $s : W \rightarrow H$ such that $s(x) = [(\gamma^{-1} \alpha)\gamma]_x$ for each $x \in W$, where $[\alpha]_c = \sigma_c \in H_c$ is any element and $[\gamma]$ is an arbitrary fixed homotopy class defines an isomorphism between H_x and H_c. Then, the change of s depends on only the change of $\sigma_c = [\alpha]_c$. Furthermore, $\varphi \circ s = 1_W$. Let us denote the totality of the mappings s defined on W by $\Gamma(W, H)$.

If B is a base for X, then $B^* = \{s(W) : W \in B,\ s \in \Gamma(W, H)\}$ is a base for H. The mappings φ and s are continuous in this topology. Moreover, φ is a locally topological mapping. Then (H, φ) is a sheaf over X. (H, φ) (or only H) is called "The Sheaf of the Fundamental Groups" over X[1]. For any open set $W \subset X$, an element s of $\Gamma(W, H)$ is called a section of the sheaf H over W. The set $\Gamma(W, H)$ is a group with the pointwise operation of multiplication. Thus, H is a sheaf of groups over X[2]. Furthermore, the group $H_x = \pi_1(X, x)$ is called the stalk of the sheaf H for any $x \in X$.

2. Characteristic Features of H[2]

2.1. Let $W \subset X$ be open set. Then, any section over W can be extended to a global section over X. Furthermore, $\Gamma(W, H) = \{s \mid W : s \in \Gamma(X, H)\} = \Gamma(X, H) \mid W$.

2.2. Any two stalks of H are isomorphic with each other.

2.3. Let $W_1, W_2 \subset X$ be any open sets, $s_1 \in \Gamma(W_1, H)$ and $s_2 \in \Gamma(W_2, H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W_1 \bigcap W_2$ then $s_1 = s_2$ over the whole $W_1 \bigcap W_2$.

2.4. Let $W \subset X$ be an open set and $s_1, s_2 \in \Gamma(W, H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W$, then $s_1 = s_2$ over the whole W.

2.5. To each point $\sigma_x \in H_x \subset H$, there uniquely corresponds a section $s \in \Gamma(W, H)$ such that $s(x) = \sigma_x$. Hence, $H_x \cong \Gamma(W, H)$. In particular, $H_x \cong \Gamma(X, H)$.

2.6. Let $x \in X$ be any point and $W = W(x)$ be an open set. Then, $\varphi^{-1}(W) = \underset{i \in I}{V}\ s_i \in \Gamma(W, H)$ and $\varphi \mid s_i(W) : s_i(W) \rightarrow W$ is a topological mapping for every $i \in I$. Hence, $W = W(x)$ is evenly covered by φ. Thus, φ is a cover projection and (H, φ) is a covering space of X. Moreover, (H, φ) is regular, because the group T of cover transformations of H is isomorphic to the group H_x, that is T is transitive on H_x [2].

3. Homomorphisms and the Group $\Gamma(X, H)$

Let X_1, X_2 be topological spaces and H_1, H_2 be the corresponding sheaves, respectively. We begin by giving the following definition.

Definition 3.1 — Let $f^* : H_1 \rightarrow H_2$ be a mapping. If f^* is continuous, a homomorphism on each stalk of H_1 and maps every stalk of H_1 into a stalk of H_2, then it is called a sheaf homomorphism.

Let $f : X_1 \rightarrow X_2$ be a continuous mapping and $f^* : H_1 \rightarrow H_2$ be a sheaf homomorphism. If $f^*(H_1)_{x_1} \subset (H_2)_{f(x_1)}$ for each $x_1 \in X_1$, then f^* is (called) a stalk preserving homomorphism with respect to f.

Definition 3.2 — Let $f^* : H_1 \rightarrow H_2$ be a sheaf homomorphism. If f^* is also a bijection, then f^* is called a sheaf isomorphism.

Theorem 3.1 — Let $f : X_1 \rightarrow X_2$ be a continuous mapping. Then there is a stalk preserving sheaf homomorphism $f^* : H_1 \rightarrow H_2$ with respect to f.

Proof : Let $x_1 \in X_1$ be any point and α be a closed path based at x_1. Then $f \circ \alpha$ is a closed path with base point $f(x_1) = x_2$ in X_2 and $[f \circ \alpha] \in (H_2)_{x_2}$. On the other hand, if α_1 and α_2 are closed paths at x_1 in X_1 such that $\alpha_1 \sim \alpha_2$, then $f \circ \alpha_1 \sim f \circ \alpha_2$. Thus, we can define the mapping $f^* : H_1 \to H_2$ such that $f^*(\sigma_{x_1}) = f^*([\alpha]_{x_1}) = [f \circ \alpha]_{f(x_1) = x_2}$ for any $[\alpha]_{x_1} = \sigma_{x_1} \in (H_1)_{x_1} \subset H_1$.

It is easily seen that the mapping f^* is well-defined, stalk preserving with respect to f and homomorphism on each stalk (Balci[1]).

To complete the proof, let us show that f^* is continuous. Let $U_2 \subset f^*(H_1) \subset H_2$ be an open set. Without loss of generality, we assume that $U_2 = s^2(W_2)$, where $W_2 \subset X_2$ is an open set and $s^2 \in \Gamma(W_2, H_2)$. Thus, $\varphi_2(U_2) = \varphi_2(s^2(W_2)) = W_2$. Since f is continuous, $f^{-1}(W_2) = W_1 \subset X_1$ is an open set. Now let $\sigma_{x_2} \in U_2$ be an element. Then, there exists at least one element $\sigma_{x_1} \in U_1 = f^{*-1}(U_2)$ such that $f^*(\sigma_{x_1}) = \sigma_{x_2}$. Since $\varphi_1(\sigma_{x_1}) = x_1 \in W_1$, there is a section $s^1 \in \Gamma(W_1, H_1)$ such that $s^1(x_1) = \sigma_{x_1}$ and $s^1(W_1) \subset H_1$ is an open set. Also $s^1(W_1) \subset U_1$. It is easily seen that $U_1 = \bigcup_{i \in I} s_i^1(W_1)$. Therefore, $U_1 \subset H_1$ is an open set, that is f^* is a continuous mapping.

Now, let C be the category of path connected topological spaces and continuous mappings and D be the category of sheaves and sheaf homomorphisms. Let us define a mapping $F : C \to D$ as follows :

$F(f) = f^* : H_1 \to H_2$ for any continuous mapping (morphism) $f : X_1 \to X_2$. Then,

(1) If $f = 1_X$, then $F(1_X) = 1_{F(X)}$, since $(1_X)^* = [1_X \circ \alpha] = [\alpha]$ for any $\sigma_x = [\alpha]_x \in H_x$.

(2) If $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_3$ are any two morphisms, then $f_2 \circ f_1 = f_2 f_1 : X_1 \to X_3$ is also a morphism and $F(f_2 f_1) = (f_2 f_1)^* : H_1 \to H_3$. However, $(f_2 f_1)^*([\alpha]) = [(f_2 f_1)\alpha]$, for any $[\alpha] \in H_{x_1} \subset H_1$. Since $(f_2 f_1)\alpha \sim f_2(f_1 \alpha)$ rel. (0, 1), it can be written that $[(f_2 f_1)\alpha] = [f_2(f_1 \alpha)] = f_2^*(f_1^*([\alpha])) = (f_2^* f_1^*)([\alpha])$.

We then have :

Theorem 3.2 — There is a covariant functor from the category of path connected topological spaces and continuous mappings to the category of sheaves and sheaf homomorphisms.

Let $f : X_1 \to X_2$ be a topological mapping, then there exists the continuous mapping $f^{-1} : X_2 \to X_1$ such that $ff^{-1} = 1_{X_2}$, $f^{-1} f = 1_{X_1}$.

From Theorem 3.1, there are the mappings $(f^{-1})^* : H_2 \to H_1$, $(ff^{-1})^* = (1_{X_2})^* : H_2 \to H_2$, $(f^{-1} f)^* = (1_{X_1})^* : H_1 \to H_1$. From Theorem 3.2, $(ff^{-1})^* = f^*(f^{-1})^* = 1_{F(X_2)}$, $(f^{-1} f)^* = (f^{-1})^* f^* = 1_{F(X_1)}$. Hence, $(f^{-1})^* = (f^*)^{-1}$. Thus, f^* is a sheaf isomorphism.

Corollary 3.1 — Let $f : X_1 \to X_2$ be a topological mapping. Then the corresponding sheaves H_1 and H_2 are isomorphic.

Let $f : (X_1, c_1) \to (X_2, f(c_1) = c_2)$ be a continuous mapping. We know that the mapping $f^* : H_1 \to H_2$ is a sheaf homomorphism. Also, each element

$\sigma_{c_1} = [\alpha]_{c_1} \in (H_1)_{c_1}$ defines a unique section s^1 over X_1 such that $s^1(x_1) = [(\gamma^{-1}\alpha)\gamma]_{x_1}$, for any $x_1 \in X_1$. However, $f^*([\alpha]_{c_1}) = [f \circ \alpha]_{c_2} \in (H_2)_{c_2}$ and $[f \circ \alpha]_{c_2}$ defines a section s^2 over X_2 such that $s^2(x_2) = [(\delta^{-1} f \circ \alpha)\delta]_{x_2}$ for any $x_2 \in X_2$. Then the correspondence $[\alpha]_{c_1} \leftrightarrow [f \circ \alpha]_{c_2}$ between $(H_1)_{c_1}$ and $(H_2)_{c_2}$ gives the correspondence $s^1 \leftrightarrow s^2$ between $\Gamma(X_1, H_1)$ and $\Gamma(X_2, H_2)$. If we denote this correspondence $f_*(s^1(x_1)) = f^*([\gamma^{-1}\alpha)\gamma]_{x_1} = [(\delta^{-1} f \circ \alpha)\delta]_{x_2} = s^2(x_2)$ then the mapping $f_* : \Gamma(X_1, H_1) \to \Gamma(X_2, H_2)$ is a homomorphism. Infact, for any two sections $s_1^1, s_2^1 \in \Gamma(X_1, H_1)$ and any point $x_2 \in X_2$,

$$f_*(s_1^1)(x_2) = f^*([(\gamma^{-1}\alpha_1)\gamma]_{x_1}) = [(\delta^{-1} f \circ \alpha_1)\delta]_{x_2},$$

$$f_*(s_2^1)(x_2) = f^*([(\gamma^{-1}\alpha_2)\gamma]_{x_1}) = [(\delta^{-1} f \circ \alpha_2)\delta]_{x_2}, \text{ and}$$

$$f_*(s_1^1) \cdot f_*(s_2^1)(x_2) = [(\delta^{-1} f \circ \alpha_1 \cdot f \circ \alpha_2)\delta]_{x_2} = [(\delta^{-1} f \circ \alpha_1 \cdot \alpha_2)\delta]_{x_2}$$

$$= f_*(s_1^1 \cdot s_2^1)(x_2).$$

We then state the following theorem.

Theorem 3.3 — Let $f : X_1 \to X_2$ be a continuous mapping. Then there exists a homomorphism $f_* : \Gamma(X_1, H_1) \to \Gamma(X_2, H_2)$.

We now give the functorial statement of this theorem. Let C be the category of path connected topological spaces and continuous mappings and D be the category of groups and homomorphisms. Let us define a mapping $F : C \to D$ with $F(X) = \Gamma(X, H)$ and $F(f) = f_*$ for any element $X \in C$ and morphism $f : X_1 \to X_2$. Then F is a covariant functor. In fact,

(1) If $f = 1_X$, then $F(1_X) = (1_X)_*$ and $(1_X)_*(s) = s$ for any $s \in \Gamma(X, H)$. Thus, $F(1_X) = 1_{F(X)}$.

(2) Let $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_3$ any morphisms. Then, $f_2 f_1 = f_2 \circ f_1 : X_1 \to X_3$ is a morphism and $F(f_2 f_1) = (f_2 f_1)_* : \Gamma(X_1, H_1) \to \Gamma(X_3, H_3)$. Moreover, $(f_2 f_1)_*(s^1) = f_{2*}(f_{1*}(s^1)) = (f_{2*} f_{1*})(s^1)$. Hence, $F(f_2 f_1) = F(f_2) F(f_1)$.

We then state the following theorem.

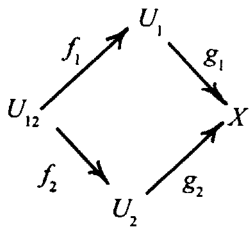
Theorem 3.4 — There is a covariant functor from the category of path connected topological spaces and continuous mappings to the category of groups and homomorphisms.

Now, let $f : X_1 \to X_2$ be a topological mapping. Then there is the mapping $f^{-1} : X_2 \to X_1$ such that $ff^{-1} = 1_{X_2}$, $f^{-1}f = 1_{X_1}$. From Theorems 3.3, 3.4, $(ff^{-1})_* = f_*(f^{-1}) = 1_{F(X_2)}$, $(f^{-1}f)_* = (f^{-1})_* f_* = 1_{F(X_1)}$. Hence, $(f^{-1})_* = (f_*)^{-1}$. Therefore f_* is an isomorphism. Notice that, for any $s^1 \in \Gamma(X_1, H_1)$ the composition $f^* \circ s^1 \circ f^{-1} \in \Gamma(X_2, H_2)$.

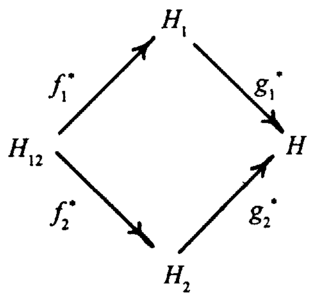
Corollary 3.2 — Let $f : X_1 \to X_2$ be a topological mapping. Then, the corresponding groups $\Gamma(X_1, H_1)$ and $\Gamma(X_2, H_2)$ are isomorphic.

4. The Seifert-Van Kampen Theorem for Global Sections [3, 4, 5, 8]

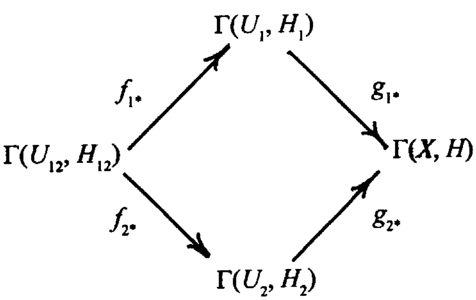
Let X be the union of the subspaces U_1 and U_2 which are both open, path connected and the intersection $U_{12} = U_1 \bigcap U_2 \neq \varphi$ and U_{12} is also path connected. Let f_1, f_2, g_1, g_2 denote various inclusion mapping as indicated below :



From Theorem 3.1, we obtain the following diagram of homomorphisms defined on the corresponding sheaves of fundamental groups.



Recall that, H_{12}, H_1, H_2 and H are the sheaves which are constructed over U_{12}, U_1, U_2 and X, respectively. Hence, we can from the following diagram of homomorphism defined on the groups of global sections.



Let us suppose that, $U_{12} = (U_{12}, c)$, $U_1 = (U_1, c)$, $U_2 = (U_2, c)$ for an arbitrary fixed point $c \in U_{12}$. Considering group presentations, assume that the groups

$\Gamma(U_{12}, H_{12}) = \langle S; R \rangle$, $\Gamma(U_1, H_1) = \langle S_1; R_1 \rangle$ and $\Gamma(U_2, H_2) = \langle S_2; R_2 \rangle$ are known. We will calculate the group $\Gamma(X, H)$ by means of these groups.

Let R_S denote the following set of words $S_1 \bigcup S_2$:

$$(f_{1*}\, s)\,(f_{2*}\, s)^{-1}, \quad s \in S.$$

We shall think of R_S as a set of relators. As a set of relations, $R_S = \{f_{1*}\, s = f_{2*}\, s : s \in S\}$. We assert that the group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_1 \bigcup S_2$ and the relations $R_1 \bigcup R_2 \bigcup R_S$. Note that the relations R of $\Gamma(U_{12}, H_{12})$ are not required. Loosely speaking $\Gamma(X, H)$ is the smallest group generated by $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ for which $f_{1*}\, s = f_{2*}\, s$, $s \in \Gamma(U_{12}, H_{12})$.

To prove this assertion we begin by giving the following lemma.

Lemma 4.1 — Let $\alpha : I \to X$ be a path and $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_n = 1$. If the mapping $\alpha_i : I \to X$ defined by $\alpha_i(t) = \alpha((1-t)t_{i-1} + tt_i)$, for $i = 1, 2, \ldots, n$ then $[\alpha] = [\alpha_1]\,[\alpha_2] \ldots [\alpha_n]$.

PROOF : The proof is by induction on n. Suppose first that $n = 2$, then $0 = t_0 \le t_1 \le t_2 = 1$ and

$$(\alpha_1 \cdot \alpha_2)(t) = \begin{cases} \alpha_1(2t), & 0 \le t \le 1/2 \\ \alpha_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2tt_1), & 0 \le t \le 1/2 \\ \alpha((1-(2t-1)\,t_1) + 2t - 1), & 1/2 \le t \le 1. \end{cases}$$

We can see that $\alpha_1 \cdot \alpha_2 \sim \alpha$ simply by using the homotopy $F : I \times J \to X$ given by

$$f(t, s) = \begin{cases} \alpha((1-s)\,2tt_1 + st), & 0 \le t \le 1/2 \\ \alpha((1-s)\,(t_1 + (2t_1 - 1)\,(1 - t_1)) + st), & 1/2 \le t \le 1. \end{cases}$$

Suppose now that $n > 2$ and the result holds for smaller integer. We have, $0 = t_0 \le t_1 \le \ldots \le t_n = 1$. Since $0 = t_0 \le t_{n-1} \le t_n = 1$ we can apply the above result to get $\alpha \sim \beta\alpha_n$, where $\beta(t) = \alpha(tt_{n-1})$. Now, $0 = \dfrac{t_0}{t_{n-1}} \le \dfrac{t_1}{t_{n-1}} \le \ldots \le \dfrac{t_{n-1}}{t_{n-1}} = 1$, so that by the inductive hypothesis, $[\beta] = [\beta_1]\,[\beta_2] \ldots [\beta_{n-1}]$, where

$$\beta_i(t) = \beta((1-t)t_{i-1}/t_{n-1} + tt_i/t_{n-1}) = \alpha((1-t)t_{i-1} + tt_i) = \alpha_i(t).$$

Thus $[\alpha] = [\alpha_1]\,[\alpha_2] \ldots [\alpha_n]$, which completes the proof.

Let us now choose the paths $q_i : I \to X$ so that $q_i(0) = c$, $q_i(1) = \alpha(t_i)$ and so that $q_i(t) \in U_{12}$ for all $t \in I$ and for $i = 1, 2, ..., n-1$. Also, let q_0 and q_n be given by $q_0(t) = q_n(t) = c$. Since $[\alpha] = [\alpha_1][\alpha_2] \ldots [\alpha_n]$, we have

$$[\alpha] = [q_0][\alpha_1][q_1^{-1}][q_1][\alpha_2][q_2^{-1}] \ldots [q_{n-1}][\alpha_n][q_n^{-1}]$$

$$= [(q_0\,\alpha_1)\,q_1^{-1}][(q_1\,\alpha_2)\,q_2^{-1}] \ldots [(q_{n-1}\,\alpha_n)\,q_n - 1]$$

and each of $q_i(\alpha_{i+1})\,q_{i+1}^{-1}$ are closed paths based c which lie entirely in U_1 or U_2. Hence $[(q_i\,\alpha_{i+1})\,q_{i+1}^{-1}]$ defines a section either in $\Gamma(U_1, H_1)$ or in $\Gamma(U_2, H_2)$ for $i = 1, 2, ..., n-1$, so that for $\lambda(k) = 1$ or 2 and for $x_{\lambda(k)} \in U_{\lambda(k)}$, $s^{\lambda(k)}(x_{\lambda(k)}) = [(\gamma^{-1}(q_i\,\alpha_{i+1})\ q_{i+1}^{-1})\,\gamma]_{x_{\lambda(k)}}$.

For brevity, let $(q_i\,\alpha_{i+1})q_{i+1}^{-1} = \delta_{i+1}$. Thus, we can write that $[\alpha] = [\delta_1][\delta_2] \ldots . [\delta_n]$ such that each $[\delta_i]$ defines a section either in $\Gamma(U_1, H_1)$ or in $\Gamma(U_2, H_2)$. Also, the homotopy class $[\alpha]$ defines a section s in $\Gamma(X, H)$, that is $s(x) = [(\gamma^{-1}\,\alpha)\gamma]_x$ for each $x \in X$. $[\alpha] = [\delta_1][\delta_2] \ldots . [\delta_n]$ implies that $s(x) = [(\gamma^{-1}\,\delta_1\,\delta_2 \ldots \delta_n)\gamma]_x$ for each $x \in X$. On the other hand, for any $x \in X$ and for $i = 1, 2, ..., n$,

$s^i(x) = [(\gamma^{-1}\,\delta_i)\gamma]_x$ and it is defined that

$$(s^i \cdot s^k)(x) = s^i(x) \cdot s^k(x), \text{ thus}$$

$$(s^1 \cdot s^2 \ldots s^n)(x) = s^1(x) \cdot s^2(x) \ldots s^n(x)$$

$$= [(\gamma^{-1}\,\delta_1)\gamma]_x \cdot [(\gamma^{-1}\,\delta_2)\gamma]_x \cdot \ldots \cdot [(\gamma^{-1}\,\delta_n)\gamma]_x$$

$$= [(\gamma^{-1}\,\delta_1\,\delta_2 \ldots \delta_n)\gamma]_x = s(x).$$

Hence, each element of $\Gamma(X, H)$ may be written as the product of images of elements from $\Gamma(U_1, H_1)$ or $\Gamma(U_2, H_2)$ under g_{1*} or g_{2*}, respectively.

Corollary 4.1 — The group $\Gamma(X, H)$ is generated by the set $g_{1*}(S_1) \bigcup g_{2*}(S_2)$ where S_1, S_2 are the generaotrs of $\Gamma(U_1, H_1)$, $\Gamma(U_2, H_2)$, respectively.

From the definition of g_{i*} we can identity S_i with $g_{i*}(S_i)$ for $i = 1, 2$. In this sense $\Gamma(X, H)$ is generated by $S_1 \bigcup S_2$ where S_1, S_2 generate $\Gamma(U_1, H_1)$, $\Gamma(U_2, H_2)$ respectively.

Lemma 4.2 — The generators of $\Gamma(X, H)$ satisfy the relations R_1, R_2 and R_S. Moreover R_1, R_2 and R_S are the unique relations in $\Gamma(X, H)$.

PROOF : Since $g_{i*} : \Gamma(U_i, H_i) \to \Gamma(X, H)$ is a homomorphism for $i = 1, 2$ any relation satisfied by the elements of S_i in $\Gamma(U_i, H_i)$ is also satisfied by the elements $g_{i*}(S_i) \subset \Gamma(X, H)$. Thus, if we use our convention of suppressing g_{i*}, the elements $S_1 \bigcup S_2$ in $\Gamma(X, H)$ satisfy the relations R_1 and R_2. If $s \in S \subset \Gamma(U_{12}, H_{12})$ then

$g_{1*} f_{1*}\, s = g_{2*} f_{2*}\, s$, since $g_1 f_1 = g_2 f_2$. If a word in S_i represents $f_{i*}\, s$, then the same word in S_i represents $g_{i*} f_{i*}\, s$ in $\Gamma(X, H)$ so that $f_{1*}\, s = f_{2*}\, s$, $s \in S$, and so the proof of the first part of Lemma 4.2 is finished.

Let us now suppose that $s = s_1^{\varepsilon(1)} s_2^{\varepsilon(2)} \ldots s_k^{\varepsilon(k)} = I$ is a relation between the elements of $S_1 \bigcup S_2 \subset \Gamma(X, H)$. Here $\varepsilon(i) = \pm 1$ and $s_i \in S_{\lambda(i)}$ for i = 1, 2, k where $\lambda(i)$ = 1 or 2. From the definition of the elements of $\Gamma(X, H)$ there is a unique element $[\alpha]$ and unique homotopy classes $[\alpha_i]$ such that $[\alpha]$ defines the sections s and each of $[\alpha_i]$ define the sections s_i. Thus, for i = 1, 2, ..., k

$$[\alpha] = [\alpha_1]^{\varepsilon(1)} [\alpha_2]^{\varepsilon(2)} \ldots [\alpha_k]^{\varepsilon(k)} = [1].$$

However, it has been proved in Godbillon[4], Massey[6] and Seifert[7] that $[\alpha]$ can be reduce to [1] by a finite sequence of operation each of which inserts or deletes an expression from a certain list. Hence s is a consequence of the relations $R_1 \bigcup R_2 \bigcup R_S$ and $R_1 \bigcup R_2 \bigcup R_S$ are the unique relations in $\Gamma(X, H)$.

As a result of Lemmas 4.1 and 4.2 we can state that,

Corollary 4.2 — The group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_1 \bigcup S_2$ and $R_1 \bigcup R_2 \bigcup R_S$.

When then state the following theorem.

Theorem 4.1 (The Seifert-Van Kampen theorem for global sections) — Let us suppose that the topological space X is the union of the subspaces U_1 and U_2 which are both open, path connected, $U_{12} = U_1 \bigcap U_2 \neq \phi$ and U_{12} is also path connected. Let the groups $\Gamma(U_{12}, H_{12})$, $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ be known. Then,

(i) (The "generators" of $\Gamma(X, H)$.) If $s \in \Gamma(X, H)$ is any section, then $s = \prod_{k=1} g_{\lambda(k)*}\, s_k$, where $s_k \in \Gamma(U_{\lambda(k)}, H_{\lambda(k)})$, $\lambda(k)$ = 1 or 2.

(ii) (The "relators" or relations" of $\Gamma(X, H)$.)

Let $s = \prod_{k=1}^{n} g_{\lambda(k)*}\, s_k \in \Gamma(X, H)$. Then $s = I$ if and only if s can be reduced to I by a finite sequence of operations each of which inserts or deletes an expression from a certain list.

If we restrict this theorem to any stalk $H_x \subset H$ for any $x \in X$, we get the known Seifert-Van Kampen Theorem at once such that it does not depend on the base point.

Theorem 4.2 (The Seifert-Van Kampen theorem for the fundamental groups) — Let us suppose that the topological space X satisfy the conditions mentioned in Theorem 4.1 and let the groups $\Gamma(U_{12}, H_{12})$, $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ be known. Then,

(i) (The "generators" of $\pi_1(X, x)$.) If $x \in X$ is any point and $[\alpha] \in \pi_1(X, x)$ is any element, then $[\alpha] = \prod_{k=1}^{n} g_{\lambda(k)*}\, s_k(x)$, where $s_k \in \Gamma(U_{\lambda(k)}, H_{\lambda(k)})$, $\lambda(k)$ = 1 or 2.

(ii) (The "relators" or "relations" of $\pi_1(X, x)$.)

Let $[\alpha] = \prod_{k=1}^{n} g_{\lambda(k)}.\, s_k(x)$. Then $[\alpha] = [1]$ if and only if $[\alpha]$ can be reduced to $[1]$ by a finite sequence of operations each of which inserts or deletes an expression from a certain list.

As a result of Theorem 4.1, and Theorem 4.2 we can state The Seifert-Van Kampen Theorem via regular covering spaces as follows :

Corollary 4.3 — Let us suppose that the topological space X is the union of the subspaces U_1 and U_2 which are both open, path connected, $U_{12} = U_1 \cap U_2 \neq \phi$ and U_{12} is also path connected. Let H, H_1, H_2 and H_{12} be corresponding sheaves of fundamental groups (which are regular covering spaces) over X, U_1, U_2 and U_{12}, respectively. Then the sheaf (regular covering space) H of fundamental groups is uniquely determined by the sheaves (regular covering spaces) H_1 and H_2 respectively.

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