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ITERATIVE METHOD FOR SOLUTIONS AND COUPLED QUASI-SOLUTIONS OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS IN ORDERED BANACH SPACES

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In this paper, we establish some new existence theorems on the solutions and the coupled minimal and maximal quasi-solutions for nonlinear Fredholm integral equations which does not possess any monotone properties in ordered Banach space by means of monotone iterative techniques and then apply these results to the two-point boundary value problem of second order nonlinear ordinary differential equations in ordered Banach spaces. Finally, a example on infinite system of nonlinear Fredholm integral equations is worked out.

1. INTRODUCTION

This paper is continuation if Sun and Liu[17]. Now we shall consider the nonlinear Fredholm integral equation

$$u(t) = \int_I H(t, s, u(s))\, ds, \quad t \in I, \qquad \ldots (1)$$

where $I = [a, b]$, $H \in C[I \times I \times E, E]$, i.e., H is a continuous mapping from $I \times I \times E$ into E, E is a real Banach space with norm $\|\cdot\|$ and there exists a function $G \in C[I \times I \times E \times E, E]$ such that for any $(t, s, x) \in I \times I \times E$

$$H(t, s, x) = G(t, s, x, x). \qquad \ldots (2)$$

Vaughan[18] established some comparison theorems and existence theorems on extremal solutions for nonlinear Volterra integral in ordered Banach space by means of monotone iterative technique. But for nonlinear Fredholm integral equations the situation is quite different since there are no comparison results in this case. Recently, in the special case where $H(t, s, x)$ is nondecreasing in x for fixed $t, s \in I$, Guo[6] establishes an existence theorem on the maximal and minimal solutions for eqn. (1) in ordered Banach spaces by means of monotone iterative techniques. The problem of proving the existence results when H does not possess any monotone assumption is an interesting and important question. The purpose of this paper is to study this problem. We shall use the Mönch's fixed point theorem[15, 3] and somewhat different method to obtain iterative sequences which converge uniformly to solutions and coupled minimal and maximal quasi-solutions of the nonlinear Fredholm integral equations in ordered Banach spaces. But, our method is different from the method of mixed monotony in Khavanin and Lakshmikantham[11] and our results generalize and improve the results of Guo[6].

2. Preliminaries and Lemmas

Let P be a cone in E that is a closed convex subset such that $\lambda P \subset P$ for any $\lambda \geq 0$ and $P \cap \{-P\} = \{\theta\}$, where θ denotes the zero element of E. By means of P a partial order $\leq$ is defined as $x \leq y$ iff $y - x \in P$. A cone P is said to be normal if there exists a constant $N > 0$ such that $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N \|y\|$ (see Deimling[3], Guo and Lakshmikantham[7]). The cone P is normal iff every ordered interval $[x, y] = \{z \in E : x \leq z \leq y\}$ is bounded. Let $P_I = \{u \in C[I, E] : u(t) \geq \theta$ for all $t \in I\}$, where $C[I, E]$ denotes the Banach space of all continuous mapping $u : I \to E$ with the norm $\|u\|_C = \max_{t \in I} \|u(t)\|$. It is clear that P_I is a cone of the space $C[I, E]$ and so it defines a partial ordering in $C[I, E]$. Obviously, the normality of P implies the normality of P_I and the normal constants of P_I and P are the same.

Let $v_0, w_0 \in C[I, E]$. Then v_0, w_0 are said to be coupled lower and upper quasi-solutions of the eqn. (1) provided

$$\left.\begin{aligned} v_0(t) &\leq \int_I G(t, s, v_0(s), w_0(s))\, ds, \\ w_0(t) &\geq \int_I G(t, s, w_0(s), v_0(s))\, ds, \quad t \in I. \end{aligned}\right\} \quad \ldots (3)$$

If in (3), equality signs hold, then v_0, w_0 are said to be coupled quasi-solutions of eqn. (1). Clearly, one can define coupled maximal and minimal quasi-solutions of eqn. (1).

We shall always assume in this paper that P is a normal cone of E. $\alpha(\cdot)$ and $\beta(\cdot)$ denote the Kuratowski's and Hausdorff's noncompactness measure, respectively, the properties of which may be found in Lakshmikantham and Leela[13]. For any $v_0, w_0 \in C[I, E]$ such that $v_0 \leq w_0$, we define the ordered interval $[v_0, w_0] = \{u \in C[I, E] : v_0 \leq u \leq w_0\}$ and the set $\Omega = \{x \in E : v_0(t) \leq x \leq w_0(t)$ for some $t \in I\}$. For any $B \subset C[I, E]$, let $B(t) = \{u(t) : u \in B\} \subset E$, $t \in I$.

The proof of our main results in this paper will need the following lemmas.

Lemma 1 — Let $G(t, s, x, y)$ be uniformly continuous on $I \times I \times \Omega \times \Omega$, $B_1, B_2 \subset [v_0, w_0]$ uniformly bounded and equicontinuous. Then for fixed $t \in I$, $G(t, s, B_1(s), B_2(s))$ is uniformly bounded and equicontinuous on $s \in I$.

The proof of Lemma 1 is simple and is therefore omitted.

Lemma 2 (Banas and Goebel[1]) — Let $B \subset C[I, E]$ be uniformly bounded and equicontinuous. Define $m(t) = \alpha(B(t))$, $t \in I$. Then $m(t)$ is continuous on $t \in I$ and

$$\alpha\left(\int_I B(s)\, ds\right) \leq \int_I \alpha(B(s))\, ds.$$

Lemma 3 (Lakshmikantham and Leela[13]) — If $B \subset C[I, E]$ is a uniformly bounded and equicontinuous, then $\alpha(B) = \max_{t \in I} \alpha(B(t))$.

Lemma 4 — Let $B_1, B_2 \subset C[I, E]$ be two countable subset satisfying $\overline{B}_1 = \overline{co}(\{u_0\} \bigcup B_2)$ for some $u_0 \in C[I, E]$. Then $\overline{B_1(t)} = \overline{co}\, (\{u_0(t)\} \bigcup B_2(t))$ for any $t \in I$.

PROOF : For any fixed $t \in I$ let $x \in B_1(t)$. Then there exists $u \in B_1$ such that $x = u(t)$. From $u \in B_1 \subset \overline{co}\, (\{u_0\} \bigcup B_2)$. We infer that there exist

$$v_n = \alpha_0^{(n)} u_0 + \sum_{k=1}^{m_n} \alpha_k^{(n)} w_k^{(n)} \in co(\{u_0\} \bigcup B_2),\ n = 1, 2, ...,$$

such that $\| v_n - u \|_C \to 0$, as $n \to \infty$, where

$$w_k^{(n)} \in B_2,\ k = 1, 2, ..., m_n, \alpha_k^{(n)} \geq 0,\ k = 0, 1, ..., m_n, \sum_{k=0}^{m_n} \alpha_k^{(n)} = 1.$$

Hence $v_n(t) \to u(t)$, as $n \to \infty$. Since

$$v_n(t) = \alpha_0^{(n)} u_0(t) + \sum_{k=1}^{m_n} \alpha_k^{(n)} w_k^{(n)}(t) \in co\ (\{u_0(t)\} \bigcup B_2(t)),\ n = 1, 2, ...,$$

we get $x = u(t) \in \overline{co}\, (\{u_0(t)\} \bigcup B_2(t))$ and so $B_1(t) \subset \overline{co}\, (\{u_0(t)\} \bigcup B_2(t))$. Therefore, $\overline{B_1(t)} \subset \overline{co}\, (\{u_0(t)\} \bigcup B_2(t))$.

Conversely, let $x \in co(\{u_0(t)\} \bigcup B_2(t))$. Then there exist $u_i \in B_2$, $i = 1, 2, ..., m$, such that

$$x = \alpha_0 u_0(l) + \sum_{i=1}^{m} \alpha_i u_i(l),$$

where $\alpha_i \geq 0$, $i = 0, 1, ..., m$ and $\Sigma_{i=0}^{m} \alpha_i = 1$. Since

$$u = \alpha_0 u_0 + \sum_{i=1}^{m} \alpha_i u_i \in co(\{u_0\} \bigcup B_2) \subset \overline{B}_1,$$

there exist $v_n \in B_1$, $n = 1, 2, ...$ such that $\| v_n - u \|_C \to 0$, as $n \to \infty$ and hence $v_n(t) \to u(t) = x$, as $n \to \infty$. From $v_n(t) \in \overline{B_1(t)}$, $n = 1, 2, ...$, we get $x \in \overline{B_1(t)}$. Thus

$co(\{u_0(t)\} \cup B_2(t)) \subset \overline{B_1(t)}$. It follows that $\overline{co}$ $(\{u_0(t)\} \cup B_2(t)) \subset \overline{B_1(t)}$. The proof is complete.

Lemma 5 (Mönch[15]) — Let X be a Banach space, $K \subset X$ closed and convex and $F: K \to K$ continuous with the property that for some $x \in K$ we have $B \subset K$ countable $\overline{B} = co$ $(\{x\} \cup F(B))$ imply B is relatively compact. Then F has a fixed point in K.

3. Main Results

Theorem 1 — Let $v_0, w_0 \in C[I, E]$ be coupled lower and upper quasi-solutions of the eqn. (1) such that $v_0 \leq w_0$. Assume that :

(H_1) G is uniformly continuous on $I \times I \times \Omega \times \Omega$;

(H_2) $G(t, s, x, y)$ is nondecreasing in $x \in \Omega$ for fixed $(t, s, y) \in I \times I \times \Omega$ and $G(t, s, x, y)$ is nonincreasing in $y \in \Omega$ for fixed $(t, s, x) \in I \times I \times \Omega$;

(H_3) there exists $k \in C[I \times I, \mathbf{R}^+]$ such that for any countable sets $D_1 \subset \Omega, D_2 \subset \Omega$ and $(t, s) \in I \times I$

$$\alpha(G(t, s, D_1, D_2)) \leq k(t, s) \max \{\alpha(D_1), \alpha(D_2)\}$$

and

$$\int_I \int_I k^2(t, s)\, ds\, dt < 1. \qquad \dots (4)$$

Then eqn. (1) has a solution $u^* \in [v^*, w^*]$, where $(v^*, w^*) \in [v_0, w_0] \times [v_0, w_0]$ is coupled minimal and maximal quasi-solutions of eqn. (1). Moreover, we have

$$v_n(t) \to v^*,\ w_n(t) \to w^*, \text{ uniformly on } t \in I \text{ as } n \to \infty,$$

where

$$\left.\begin{aligned} v_n(t) &= \int_I G(t, s, v_{n-1}(s), w_{n-1}(s))\, ds, \\ w_n(t) &= \int_I G(t, s, w_{n-1}(s), v_{n-1}(s))\, ds, \quad n = 1, 2, 3, \dots, t \in I \end{aligned}\right\} \qquad \dots (5)$$

and

$$\begin{aligned} v_0(t) \leq v_1(t) \leq \dots \leq v_n(t) \leq \dots \leq v^*(t) \leq u^*(t) \\ \leq w^*(t) \leq \dots \leq w_n(t) \leq \dots \leq w_1(t) \leq w_0(t), \quad t \in I. \end{aligned}$$

Proof : We first define the operator $A : [v_0, w_0] \times [v_0, w_0] \to C[I, E]$ by the formula

$$A(v, w) = \int G(t, s, v(s), w(s))\, ds. \qquad \dots (6)$$

It follows from the assumption (H_2) that A is a mixed monotone operator, i.e., $A(v, w)$ is nondecreasing in $v \in [v_0, w_0]$ and nonincreasing in $w \in [v_0, w_0]$. Define the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ by (5). Since A is a mixed monotone operator and $v_0 \leq A(v_0, w_0), A(w_0, v_0) \leq w_0$, it is easy to see that

$$v_0(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq w_n(t) \leq \ldots \leq w_1(t) \leq w_0(t), \quad t \in I.$$

Further, by the normality of P_I, we deduce that $[v_0, w_0]$ is bounded in $C[I, E]$. Therefore, by the assumption (H_1), A is a continuous and bounded from $[v_0, w_0] \times [v_0, w_0]$ into $[v_0, w_0]$. Now, for any countable subsets $B_1, B_2 \subset \overline{co}\, A([v_0, w_0] \times [v_0, w_0]) \subset [v_0, w_0]$, by the uniform continuity of G on $I \times I \times \Omega \times \Omega$ and the normality of P_I, it is easy to prove that $\overline{co}\, A([v_0, w_0] \times [v_0, w_0])$ is uniformly bounded and equicontinuous.

Hence, by the Lemma 1, for fixed $t \in I$, $G(t, s, B_1(s), B_2(s))$ is uniformly bounded and equicontinuous on $s \in I$. It follows from Lemma 2 and the assumption (H_3) that

$$\begin{aligned} \alpha(A(B_1(t), B_2(t))) &\leq \int_I \alpha(G(t, s, B_1(s), B_2(s)))\, ds \\ &\leq \int_I k(t, s) \max\ \{\alpha(B_1(s)), \alpha(B_1(s))\} ds, \quad t \in I. \qquad \ldots (7) \end{aligned}$$

In particular, for uniformly bounded and equicontinuous countable subsets

$$B_1 = \{v_n : n = 1, 2, 3, \ldots\},$$

$$B_2 = \{w_n : n = 1, 2, 3, \ldots\} \subset \overline{co}\, A([v_0, w_0] \times [v_0, w_0]),$$

we have

$$\begin{cases} \alpha(B_1(t)) = \alpha(\{A(v_n(t), w_n(t)) : n = 0, 1, 2, \ldots.\}) \leq \alpha(A(B_1(t), B_2(t))), \\ \alpha(B_2(t)) = \alpha(\{A(w_n(t), v_n(t)) : n = 0, 1, 2, \ldots.\}) \leq \alpha(A(B_2(t), B_1(t))), \ t \in I, \end{cases}$$

and so, by (7)

$$\max\{\alpha(B_1(t)), \alpha(B_2(t))\} \leq \int_I k(t, s) \max\ \{\alpha(B_1(s)), \alpha(B_2(s))\} ds, \ t \in I.$$

Set $m(t) = \max\{\alpha(B_1(t)), \alpha(B_2(t))\}$, $t \in I$. Then $m(t) \leq \int_I k(t, s)\, m(s) ds$, $t \in I$ and therefore

$$\begin{aligned} \int_I m^2(t) dt &\leq \int_I \left(\int_I k(t, s) m(s) ds \right)^2 dt \\ &\leq \int_I \left(\int_I k^2(t, s) ds \int_I m^2(s) ds \right) dt \\ &= \left(\int_I \int_I k^2(t, s) ds\, dt \right) \left(\int_I m^2(s) ds \right), \end{aligned}$$

which, by virtue of (4) and continuity of $m(t)$, implies $m(t) = 0$, $t \in I$, i.e., $\alpha(B_1(t)) = \alpha(B_2(t)) = 0$, $t \in I$. Consequently, by Lemma 3. $\alpha(B_1) = \alpha(B_2) = 0$, that is, $\{v_n\}$ and $\{w_n\}$ are relatively compact on $C[I, E]$, and so there exist subsequences $\{v_{n_k}\} \subset \{v_n\}$ and $\{w_{n_k}\} \subset \{w_n\}$ such that

$$v_{n_k} \to v^* \in [v_0, w_0], \quad w_{n_k} \to w^* \in [v_0, w_0], \quad k \to \infty.$$

Since P_I is normal and $\{v_n\}$ and $\{w_n\}$ are monotone sequences, it is easy to see that the sequences $\{v_n\}$ and $\{w_n\}$ also converge uniformly to v^* and w^*, respectively. Taking limit in (5), we find that (v^*, w^*) is a coupled quasi-solutions of the eqn. (1). Let $(v, w) \in [v_0, w_0] \times [v_0, w_0]$ be any coupled quasi-solutions of eqn. (1). It follows easily from the mixed monotone property of A and $v_0 \le v, w \le w_0$ that $v_1 \le v, w \le w_1$. By induction, it is easy to see that

$$v_n \le v, w \le w_n, n = 1, 2, 3, \dots \quad \dots (8)$$

Letting $n \to \infty$ in (8), we get $v^* \le v, w \le w^*$. This shows that (v^*, w^*) is the coupled minimal and maximal quasi-solutions of the eqn.(1).

Finally, we prove that eqn. (1) has a solution $u^* \in [v^*, w^*]$. Define a continuous operator $F : [v^*, w^*] \to [v_0, w_0]$ by $Fu = A(u, u)$. It is evident that $u \in [v^*, w^*]$ is a solution of eqn. (1) iff u is a fixed point of F in $[v^*, w^*]$. For any $u \in [v^* w^*]$, by the mixed monotone property of A, we have $v^* = A(v^*, w^*) \le Fu \le A(v^*, w^*) = w^*$, hence $F : [v^*, w^*] \to [v^*, w^*]$. Set $K = \overline{co}\,(F[v^*, w^*])$. Then F is a continuous operator from K into K. Let $B \subset K \subset C[I, E]$ be any countable subset satisfying $\overline{B} = \overline{co}\,(\{x\} \cup F(B))$ for some $x \in K$ by Lemma 4 for any $t \in I$, we get $\overline{B(t)} = \overline{co}\,(\{x(t)\} \cup (FB)(t))$ and from (7) we obtain

$$\alpha(B(t)) = \alpha\,(\,\overline{B(t)}\,) \le \alpha((FB)(t))$$

$$= \int_I \alpha(G(t, s, B)(s), B(s)))ds \le \int_I K(t, s)\ \alpha\ (B(s))ds.$$

Using the same method as above, by (4) we assert that $\alpha(B(t)) = 0$, $t \in I$. Consequently, by Lemma 3, we have $\alpha(B) = 0$, i.e., B is relatively compact in the space $C[I, E]$. It follows from Lemma 5 that F has a fixed point u^* in $K = \overline{co}\,(F[v^*, w^*]) \subset [v^*, w^*]$, i.e., u^* is a solution of eqn. (1) and $v^* \le u^* \le w^*$. This completes the proof of Theorem 1.

Remark 1 : Observe that, the conclusions of Theorem 1 cannot be obtained by Theorem 1 of Gou and Lakshmikantham[9], and Theorem 1 of Chen[2].

Remark 2 : Obviously, suppose that H of eqn. (1) admits a decomposition of the form $H(t, s, x) = H_1(t, s, x) + H_2(t, s, x)$, where $H_1(t, s, x)$ is nondecreasing in x for fixed $(t, s) \in I \times I$ and $H_2(t, s, x)$ is nonincreasing in x for fixed $(t, s) \in I \times I$.

Then $G(t, s, x, y) = H_1(t, s, x) + H_2(t, s, y)$ satisfies condition (H_2) of Theorem 1. However, $H(t, s, x)$ does not possess any monotone properties in x.

Remark 3 : Suppose that there exist $v_0, w_0 \in C[I, E]$, $v_0 \leq w_0$, and a positive bounded linear mapping L from E into E such that

$$\begin{cases} v_0(t) \leq \int_I (H(t, s, v_0(s)) + H(t, s, w_0(s)) - L(w_0(s) - v_0(s)))\, ds, \\ \\ w_0(t) \geq \int_I (H(t, s, w_0(s)) + H(t, s, v_0(s)) + L(w_0(s) - v_0(s)))\, ds,\ t \in I, \end{cases}$$

and for any $t, s \in I, x, y \in \Omega, x \leq y$

$$-L(y-x) \leq H(t, s, y) - H(t, s, x) \leq L(y-x).$$

Then $G(t, s, x, y) = [H(t, s, x) + H(t, s, y) + L(x-y)]/2$ satisfies condition (H_2) of Theorem 1 and v_0, w_0 are coupled lower and upper quasi-solutions of eqn. (1).

The proof is simple and we omit the details.

Theorem 2 — Let $v_0, w_0 \in C[I, E]$ be coupled lower and upper quasi-solutions of the eqn.(1) such that $v_0 \leq w_0$. Assume that the assumption (H_2) of Theorem 1 holds.

Suppose further that there exists $k \in C[I \times I, \mathbf{R}^+]$ such that for any $(t, s, x, y) \in I \times I \times \Omega \times \Omega, x \leq y$

$$G(t, s, y, x) - G(t, s, x, y) \leq k(t, s)\,(y-x)$$

and $\rho(K) < 1$, where $\rho(K)$ denotes the spectral radius of linear operator $(Ku)(t) = \int_I k(t, s)\, u(s) ds$. Then eqn. (1) has exactly one solution $u^* \in [v_0, w_0]$ and for any initial $(p_0, q_0) \in [v_0, w_0] \times [v_0, w_0]$, constructing successively sequences

$$\left. \begin{aligned} p_n(t) &= \int_I G(t, s, p_{n-1}(s), q_{n-1}(s))\, ds, \\ q_n(t) &= \int_I G(t, s, q_{n-1}(s), p_{n-1}(s))\, ds,\ n = 1, 2, 3, \ldots, t \in I \end{aligned} \right\} \quad \ldots (9)$$

we have

$$p_n(t) \to u^*,\ q_n(t) \to u^*, \text{ uniformly on } t \in I \text{ as } n \to \infty. \quad \ldots (10)$$

In particular, for any initial $u_0 \in [v_0, w_0]$, the successively sequences $\{u_n(t)\}$ defined by

$$u_n(t) = \int_I G(t, s, u_{n-1}(s), u_{n-1}(s))\, ds,\ n = 1, 2, 3, \ldots, t \in I,$$

converge uniformly to unique solution $u^* \in [v_0, w_0]$ of eqn. (1).

PROOF : It follows from the assumptions that $A : [v_0, w_0] \times [v_0, w_0] \to C[I, E]$ defined by (6) is a mixed monotone operator such that

$$v_0 \leq A(v_0, w_0),\ A(w_0, v_0) \leq w_0, \qquad \text{... (11)}$$

$$A(w, v) - A(v,w) \leq K(w - v) \text{ whenever } w, v \in [v_0, w_0], v \leq w. \qquad \text{... (12)}$$

Define $\{v_n(t)\}$ and $\{w_n(t)\}$ by (5). It follows from (11) and (12) that

$$v_0(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq w_n(t) \leq \ldots \leq w_1(t) \leq w_0(t),\ t \in I,$$

$$\theta \leq w_n(t) - v_n(t) \leq K^n (w_0(t) - v_0(t)),\ t \in I, n = 1, 2, 3, \ldots, \qquad \text{... (13)}$$

and consequently

$$\theta \leq v_{n+k}(t) - v_n(t) \leq w_n(t) - v_n(t) \leq K^n (w_0(t) - v_0(t)),$$

$$t \in I, k, n = 1, 2, 3, \ldots . \qquad \text{... (14)}$$

By virtue of the normality of P_I and (14), there exists a constant $N > 0$ such that

$$\| v_{n+k} - v_n \|_C \leq N \| K^n \| \| w_0 - v_0 \|_C, k, n = 1, 2, 3, \ldots$$

which, by virtue of $\rho(K) < 1$, implies that $\{v_n\}$ is a Cauchy sequence in $C[I, E]$, also so it converges uniformly to some $u^* \in [v_0, w_0]$. By (13), we can prove that $\{w_n\}$ also converges uniformly to u^* and $v_n \leq u^* \leq w_n$. Hence, from the mixed monotone property of A, we have.

$$v_n = A(v_{n-1}, w_{n-1}) \leq A(u^*, u^*) \leq A(w_{n-1}, v_{n-1}) = w_n, n = 1, 2, 3, \ldots .$$

$$\text{... (15)}$$

Letting $n \to \infty$ in (15), we get $u^* = A(u^*, u^*)$. Let $u \in [v_0, w_0]$ be any a fixed point of A. Since A is mixed monotone and $v_0 \leq u \leq w_0$, it is easy to see that

$$v_1 = A(v_0, w_0) \leq A(u, u) = u \leq A(w_0, v_0) = w_1.$$

By induction, it is easy to see that

$$v_n \leq u \leq w_n, n = 1, 2, 3, \ldots . \qquad \text{... (16)}$$

Letting $n \to \infty$ in (16), we get $u = u^*$, i.e., we have showed that u^* is the unique solution of eqn. (1).

Finally let $(p_0, q_0) \in [v_0, w_0] \times [v_0, w_0]$ be given and (9) be constructed. Similarly the proof of (8), we get

$$v_n \leq p_n, q_n \leq w_n, n = 0, 1, 2, \ldots . \qquad \text{... (17)}$$

It follows therefore from (17) and the normality of P_I that (10) holds. Thus the proof is complete.

Theorem 3 — Let H be a continuous mapping from $I \times I \times P \times P$ into P. Assume that the following conditions are satisfied :

(G_1) $G(t, s, x, y)$ is nondecreasing in $x \in P$ for fixed $(t, s, y) \in I \times I \times P$ and $G(t, s, x, y)$ is nonincreasing in $y \in P$ for fixed $(t, s, x) \in I \times I \times P$;

(G_2) there exists $0 < \alpha < 1$ such that for any $(t, s, x, y) \in I \times I \times P \times P$, $0 < \lambda < 1$

$$G\left(t, s, \lambda x, \frac{y}{\lambda}\right) \geq \lambda^{\alpha} G(t, s, x, y);$$

(G_3) there exist $u_0 \in P_I$ and $\lambda_0 \in (0, 1)$ such that

$$\lambda_0^{(1-\alpha)/2} u_0(t) \leq \int_I H(t, s, u_0(s)) ds \leq \lambda_0^{(\alpha-1)/2} u_0(t), \; t \in I.$$

Then eqn. (1) has exactly one solution $u^* \in [v_0, w_0]$, where $v_0 = \lambda_0^{1/2} u_0$, $w_0 = \lambda_0^{-1/2} u_0$. Moreover, for any initial points $(p_0, q_0) \in [v_0, w_0] \times [v_0, w_0]$ constructing successively sequences (9) we have (10) with convergence rate

$$\| p_n - u^* \|_C = O(1 - \lambda_0^{\alpha^n}), \; \| q_n - u^* \|_C = O(1 - \lambda_0^{\alpha^n}).$$

PROOF: Define operator $A : P_I \times P_I \rightarrow P_I$ by (6). It follows from the assumptions (G_1)-(G_3) that A is mixed monotone and

$$A\left(\lambda v, \frac{w}{\lambda}\right) \geq \lambda^{\alpha} A(v, w), \; v, w \in P_I, \; 0 < \lambda < 1,$$

$$\lambda_0^{(1-\alpha)/2} u_0 \leq A(u_0, u_0) \leq \lambda_0^{(\alpha-1)/2} u_0.$$

Hence, using the similar method as in the proof of Theorem 1 in Guo[5], we assert that the conclusions of Theorem 3 hold. We omit the details.

Remark 4 : It should be pointed out that we do not require the condition (G_3) when P is a solid cone.

Remark 5 : In Theorem 1 we use the compactness type condition (H_3). But in Theorem 2 and Theorem 3 of this paper, we do not use the compactness type conditions.

4. APPLICATIONS

Consider the following two-point BVP

$$\left.\begin{aligned} -u'' &= f(t, u), \; t \in I = [0, 1], \\ u(0) &= u(1) = 0, \end{aligned}\right\} \quad \ldots (18)$$

where $f \in C[I \times P, P]$ and P is a cone in a real Banach space E.

Theorem 4 — Let P be normal. Suppose that there exists a mapping $g \in C[I \times P \times P, P]$ such that $f(t, x) = g(t, x, x)$ and g satisfies the following conditions :

(Q_1) g is uniformly continuous on $I \times P_R \times P_R$ for any $R > 0$, where $P_R = \{x \in P : \| x \| \le R\}$;

(Q_2) $g(t, x, y)$ is nondecreasing in $x \in P$ for fixed $(t, y) \in I \times P$ and $g(t, x, y)$ is nonincreasing in $y \in P$ for fixed $(t, x) \in I \times P$;

(Q_3) there eixst a nonnegative constant $c < 8$ and $d(t) \in C[I, P]$ such that

$$g(t, x, \theta) \le cx + d(t), t \in I, x \in P;$$

(Q_4) there exists a constant $0 < M < \sqrt{90}$ such that for any bounded and countable subsets $D_1 \subset P, D_2 \subset P$ and $t \in I$

$$\alpha(g(t, D_1, D_2)) \le M \max \{\alpha(D_1), \alpha(D_2)\}.$$

Then there is a positive function $w_0 \in C[I, E]$ such that the eqn.(1) has a coupled minimal and maximal quasi-solutions $(v^*, w^*) \in [\theta, w_0] \times [\theta, w_0]$ and a solution $u^* \in [v^*, w^*]$. Moreover, there exist sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly and monotonically to v^* and w^*, respectively.

PROOF : It is well known that u is a solution of BVP (18) in $C^2[I, P]$ iff u is a solution in $C[I, P]$ of the following integral equation

$$u(t) = \int_I h(t, s) f(s, u(s))ds, \quad \text{... (19)}$$

where

$$h(t, s) = \begin{cases} t(1-s), & \text{if } t \le s, \\ s(1-t), & \text{if } t > s. \end{cases}$$

It is easy to verify that the mapping $G(t, s, x, y) = h(t, s)\, g(s, x, y)$ satisfies the assumption (H_1) and (H_2) of Theorem 1. For any bounded and countable subsets $D_1 \subset P, D_2 \subset P$ and $t, s \in I$ by the assumption (Q_4), we have

$$\begin{aligned}\alpha(G(t, s, D_1, D_2)) &= \alpha(h(t, s,)\, g(s, D_1, D_2)) \\ &\le h(t, s)\, \alpha(g(s, D_1, D_2)) \\ &\le h(t, s) M \max \{\alpha(D_1), \alpha(D_2)\}\end{aligned}$$

and

$$\int_I \int_I (Mh(t, s))^2\, ds\, dt = M^2 \int_I \int_I h^2(t, s)\, ds\, dt = \frac{M^2}{90} < 1.$$

Hence the assumption (H_3) of Theorem 1 is satisfied for $k(t, s) = Mh(t, s)$. Put

$$Lw(t) = c \int_I h(t, s) w(s) ds, \quad y_0(t) = \int_I h(t, s) d(s) ds, \ t \in I.$$

Since $\| L \| \le c \max_{t \in I} t(1-t)/2 = c/8 < 1$, the equation $(I - L)\, w = y_0$ has a unique solution $w_0(t) = (I - L)^{-1} y_0 = \sum_{n=0}^{\infty} L^n y_0 \in P_I$. Hence, by the assumption (Q_3), for any $t \in I$ we obtain

$$\int_I h(t, s)g(s, w_0(s), \theta)ds \le \int_I h(t, s)(cw_0(s) + d(s))ds$$

$$= Lw_0(t) + y_0(t) = w_0(t)$$

and

$$\theta \le \int_I G(t, s)g(s, \theta, w_0(s))ds.$$

Therefore $(\theta, w_0(t))$ is a coupled quasi-solutions of eqn. (19). The conclusion of Theorem 4 follows from Theorem 1

Example — Consider the infinite system of nonlinear integral equations of Fredholm type

$$u_n(t) = n^{-1/2} \int_I t\cos(ts)\, u_n(s)\left(1 + u_{n+1}^2(s)\right)^{-1} ds$$

$$+ 4n^{-1} \int_I t(1 + t + s)^{-3} u_{2n}^2(s)\sin(ts)ds$$

$$+ 0.3n^{-2} \int_I t^2 s e^{u_n(s)} ds, \quad t \in I = [0, 1], \; n = 1, 2, 3, \ldots \ldots \qquad (20)$$

Conclusion — The system (20) has a continuous solution

$$u^* = (u_1^*(t), u_2^*(t), \ldots, u_n^*(t), \ldots), t \in I$$

and a coupled minimal and maximal continuous quasi-solutions

$$v^*(t) = (v_1^*(t), v_2^*(t), \ldots, v_n^*(t), \ldots), w^*(t) = (w_1^*(t), w_2^*(t), \ldots, w_n^*(t), \ldots), t \in I$$

satisfying

$$\theta \le v_n^*(t) \le u_n^*(t) \le w_n^*(t) \le \frac{t}{n}, \quad t \in I, n = 1, 2, 3, \ldots .$$

Moreover, there exist sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly and monotonically to v^* and w^*, respectively.

PROOF : Let $E = c_0 = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \to 0\}$ with norm $\| x \| = \sup_n \| x_n \|$ and $P = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in c_0 : x_n \ge 0$ for all $n\}$. Then P is a normal cone in c_0 and the system (20) may be regarded as an equation of the form (1), where

$$x = (x_1, x_2, \ldots, x_n, \ldots), y = (y_1, y_2, \ldots, y_n, \ldots) \in c_0,$$

$$H(t, s, x) = G(t, s, x, x),$$

$$G(t, s, x, y) = (G_1(t, s, x, y), G_2(t, s, x, y), \ldots, G_n(t, s, x, y), \ldots)$$

and

$$G_n(t, s, x, y) = n^{-1/2}\, t \cos(ts) x_n (1 + y_{n+1}^2)^{-1}$$
$$+ 4n^{-1} t(1 + t + s)^{-3} x_{2n}^2 \sin(ts)$$
$$+ 0.3n^{-2} t^2 s e^{x_n},\ n = 1, 2, 3, \ldots . \qquad \ldots (21)$$

Evidently, $G \in C[I \times I \times c_0 \times c_0, c_0]$, G is uniformly continuous on $I \times I \times B_R \times B_R$ for any $R > 0$, where $B_R = \{x \in c_0 : \| x \| \le R\}$, and $G(t, s, x, y)$ is nondecreasing in $x \in P$ for fixed $(t, s, y) \in I \times I \times P$ and nonincreasing in $y \in P$ for fixed $(t, s, x) \in I \times I \times P$. Moreover, (21) implies

$$| G_n(t, s, x, y) | \le n^{-1/2} \| x \| + 4n^{-1} \| x \|^2$$
$$+ 0.3n^{-2} e^{\| x \|},\ n = 1, 2, 3, \ldots . \qquad \ldots (22)$$

Now consider the sequences $\{x^{(m)}\}$, $\{y^{(m)}\} \subset P$ and let $\max\{\| x^{(m)} \|, \| y^{(m)} \|\} \le M =$ const. $(m = 1, 2, 3, \ldots)$. Taking into account (22) and using the diagonal method, we can choose subsequences $\{x_m^{(m)}\} \subset \{x^{(m)}\}$ and $\{y_m^{(m)}\} \subset \{y^{(m)}\}$ such that

$$G_n\left(t, s, x_m^{(m)}, y_m^{(m)}\right) \to z_n \text{ as } m \to \infty, n = 1, 2, 3, \ldots , \qquad \ldots (23)$$

where $t, s \in I$ are fixed. By (22), we have

$$\left| G_n\left(t, s, x_m^{(m)}, y_m^{(m)}\right) \right| \le n^{-1/2} M + 4n^{-1} M^2 + 0.3n^{-2} e^M \qquad \ldots (24)$$

$$| z_n | \le n^{-1/2} M + 4n^{-1} M^2 + 0.3n^{-2} e^M, n, m = 1, 2, 3, \ldots , \qquad \ldots (25)$$

and so, for any given $\varepsilon > 0$, we can choose a positive integer n_0 sufficiently large such that

$$\left| G_n\left(t, s, x_m^{(m)}, y_m^{(m)}\right) \right| < \frac{\varepsilon}{2}, | z_n | < \frac{\varepsilon}{2}, n > n_0\ m = 1, 2, 3, \ldots . \qquad \ldots (26)$$

By virtue of (23), there exists a positive integer m_0 such that

$$\left| G_n\left(t, s, x_m^{(m)}, y_m^{(m)}\right) - z_n \right| < \varepsilon, m > m_0,\ n = 1, 2, \ldots, n_0. \qquad \ldots (27)$$

It follows from (26) and (27) that

$$\left\| G\left(t, s, x_m^{(m)}, y_m^{(m)}\right) - z \right\| = \sup_n \left| G_n\left(t, s, x_m^{(m)}, y_m^{(m)}\right) - z_n \right| < \varepsilon, m > m_0,$$

where $z = (z_1, z_2, \ldots, z_n, \ldots) \in c_0$ on account of (25). This means that $G\left(t, s, x_m^{(m)}, y_m^{(m)}\right) \to z$ in c_0 as $m \to \infty$ and therefore we have proved that $G(t, s, D_1, D_2)$ is relatively compact in $E = c_0$ for any bounded and countable subset $D_1 \subset P, D_2 \subset P$ and $t, s \in I$. Hence, the assumption (H_3) of Theorem 1 is satisfied for $k(t, s) \equiv 0$.

Now let $v_0(t) = (0, 0, ..., 0, ...)$ and $w_0(t) = (t, t/2, \ldots, t/n, ...)$ for $t \in I$. Evidently, $v_0, w_0 \in C[I, E]$ satisfy the inequality

$$v_0(t) \leq \int_I G(t, s, v_0(s), w_0(s))ds, \; t \in I.$$

It is not difficult to show that v_0 and w_0 also satisfy the inequality

$$w_0(t) \geq \int_I G(t, s, w_0(s), v_0(s))ds, \; t \in I. \qquad \text{... (28)}$$

In fact, we have

$$\int_I s^2 t(1 + t + s)^{-3} \sin(ts)ds \leq t^2 \int_I s^3 \, ds \leq \frac{t}{4},$$

$$\int_I t^2 \, se^{s/n} \, ds \leq t \int_I e^{s/n} \, ds = tn\left(e^{1/n} - 1\right) \leq 0.8tn,$$

and consequently

$$\int_I G_n(t, s, w_0(s), v_0(s))ds = n^{-1/2} \int_I t \cos(ts)\left(\frac{s}{n}\right) ds$$

$$+ 4n^{-1} \int_I t(1 + t + s)^{-3} \left(\frac{s}{2n}\right)^2 \sin(ts) \, ds$$

$$+ 0.3n^{-2} \int_I t^2 \, se^{s/n} \, ds \leq \frac{t}{2n} + \frac{t}{4n} + \frac{t}{4n}$$

$$= \frac{t}{n}, \; t \in I, n = 1, 2, 3, \ldots .$$

This means that (28) holds and (v_0, w_0) is a coupled quasi-solutions of the eqn.(1). Hence, our required conclusions follow from Theorem 1.

Remark 6 : The results of this paper cannot be obtained by the fixed point theorems on increasing operators and decreasing operators, since H in the eqn.(1) does not possess any monotone properties. Of course in Theorem 1 and Theorem 4 one does not need the compactness type conditions when $E = \mathbf{R}^n$.

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