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NON-HYPONORMAL WEIGHTED COMPOSITION OPERATORS

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In this paper, weighted composition operators of class (M, k), $k \geq 2$ on L^2-spaces are characterized and their various properties are studied.

1. PRELIMINARIES

Let (X, Σ, λ) be a σ-finite measure space. A transformation T on (X, Σ) is a Σ-measurable mapping from X onto X such that $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ. A weighted composition operator is a linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$Wf = wf \circ T$$

where w is a complex valued, Σ measurable function. In case $w = 1$ a.e., W becomes a composition operator, denoted by C_T.

To examine the weighted composition operators efficiently, Lambert[3], associated with each transformation T, the so-called conditional expectation operator $E(./T^{-1}\Sigma) = E(\,.\,)$. $E(f)$ is defined for each non-negative measurable function f or for each $f \in L^{\rho}$ $(1 \leq \rho)$, and is uniquely determined by the conditions :

(i) $E(f)$ is $T^{-1}\Sigma$ measurable and

(ii) If B is any $T^{-1}\Sigma$ measurable set for which $\int_B f\,d\lambda$ converges we have

$$\int_B f\,d\lambda = \int_B E(f)\,d\lambda.$$

As an operator on L^{ρ}, E is the projection onto the closure of the range of C_T. E is the identity on L^{ρ} if and only if $T^{-1}\Sigma = \Sigma$.

For each f there is a Σ-measurable function F such that $E(f/T^{-1}\Sigma) = F \circ T$. Moreover, if G is another such function then $F = G$ a.e. on supp h. Thus $h \,.\, E(f/T^{-1}\Sigma) \circ T^{-1}$ is well-defined even when T is not invertible[3].

We will be concerned with the operators acting on $L^2 = L^2(X, \Sigma, \lambda)$.

The Radon-Nikodym derivative of λT^{-1} with respect to λ is denoted by h and that of $\lambda \circ T^{-k}$ with respect to λ is denoted by h_k where T^k is obtained by composing $T - k$ times. Let w_k denote $w(w_0 T)(w_0 T^2) \ldots (w_0 T^{k-1})$ so that $W^k f = w_k (f \circ T^k)$.

A quasi-normal operator T on a Hilbert space H is one for which $T(T^* T) = (T^* T)T$. Clearly, for such operators, $T^{*k} T^k = (T^* T)^k$ for $k \geq 2$. This fact provides a motivation to generalise the class of quasi-normal operators as follows : An operator T is defined to be of class $(M\,;\ k)$ if $T^{*k} T^k \geq (T^* T)^k, k \geq 2$. Obviously $(M\,;\ 2)$ class contains hyponormal operators. However, the class $(M;\ k)$, $k > 2$ does not include all hyponormal operators[5].

T is M-paranormal if for all unit vectors $x \in H$, $\| Tx \|^2 \leq M \| T^2 x \|$ and

T is M-quasi-hyponormal if there exists $M > 0$ such that

$$M^2 T^{*2} T^2 - (T^* T)^2 \geq 0.$$

When $M = 1$, these operators are respectively called paranormal and quasi-hyponormal (or of $(M;\ 2)$ class). We have the following inclusion relations which are known to be proper.

$$\{\text{Quasinormal operators}\} \subseteq \{\text{Hyponormal operators}\}$$

$$\subseteq \{(M;\ 2)\ \text{Class operators}\} \subseteq \{\text{Paranormal operators}\}.$$

In Whitley[7], the quasinormal C_T are characterized $(h = h \circ T)$ and in Campbell *et al.*[1] the quasinormal W are characterized $(J = J \circ T$ on $\sigma(w))$.

In Campbell and Horner[2], it is shown that W^* is quasinormal iff $\Sigma_{\sigma(J)} \subseteq (T^{-1}(\Sigma))_{\sigma(w)}$ and $hE(w^2) \circ T^{-1} = wE(w)\, h \circ T$ on $\sigma(J)$.

In Lambert[3], hyponormal C_T are characterized ($h > 0$ a.e. and $h \circ T\, E(1/h) \leq 1$ a.e.) and hyponormal W are also characterized.

$$(\sigma(w) \subseteq \sigma(J) \text{ and } h \circ T[E(w^2/J)] \leq \Psi_{\sigma(E(w))} \text{ a.e.}].$$

In Veluchamy and Panayappan[6], it is shown that C_T is M-paranormal iff C_T is M-quasihyponormal iff $M^2 g_0 \geq f_0$ a.e. In Panayappan[4], C_T of class $(M;\ k)$ are characterized. $[f_0^k \leq f_0^{(k)}$ a.e.].

The aim of this paper is to generalise the results obtained for the composition operators in an earlier paper by the author[4] to the weighted composition operators.

2. Main Results

Theorem 2.1 — W is of class $(M,\ k)$ if and only if $h_k E(w_k^2) \circ T^{-k} \geq h^K [E(w^2) \circ T^{-1}]^k$ a.e.

Proof : We have, $W^k f = w_k (f \circ T^k)$

and

$$W^{*}f = h_k E(w_k f) \circ T^{-k}$$

and so

$$W^{*K} W^{k} f = W^{*K} [w_k (f \circ T^{k})]$$

$$= h_k E(w_k^2 f \circ T^{k}) \circ T^{-k}$$

$$= h_k E(w_k^2) \circ T^{-k} f.$$

Also

$$W^{*} W f = hE(w^2) \circ T^{-1} f.$$

Then

W is of class (M, k)

$\Leftrightarrow W^{*k} W^{k} \geq (W^{*} W)^{k}$

$\Leftrightarrow \langle [W^{*K} W^{K} - (W^{*} W)^{K}] f, f \rangle \geq 0 \quad$ for every $f \in L^2$

$\Leftrightarrow \int_E h_K E(w_K^2) \circ T^{-K} - [hE(w^2) \circ T^{-1}]^{K} |f|^2 \, d\lambda \geq 0$

for every $E \in \Sigma$

$\Leftrightarrow h_K E(w_K^2) \circ T^{-K} \geq h^{K} [E(w^2) \circ T^{-1}]^{K}$ a.e.

Corollary 2.2 — W is M-quasihyponormal if and only if

$$M^2 h_2 E(w_2^2) \circ T^{-2} \geq h^2 [E(w^2) \circ T^{-1}]^2 \text{ a.e.}$$

PROOF : W is M-quasihyponormal if and only if

$$M^2 W^{*2} W^2 - (W^{*} W)^2 \geq 0.$$

Assume $K = 2$ in Theorem 2.1.

Theorem 2.3 — W is M-paranormal if and only if

$$M^2 h_2 E(w_2^2) \circ T^{-2} \geq h^2 [E(w^2) \circ T^{-1}]^2 \text{ a.e.}$$

PROOF : By Theorem (B) of Veluchamy and Panayappan[6], W is M-paranormal if and only if

$$M^2 W^{*2} W^2 + 2K W^{*} W + K^2 \geq 0 \text{ for all } K \in R.$$

Hence

$$M^2 h_2 E(w_2^2) \circ T^{-2} + 2K h E(w^2) \circ T^{-1} + K^2 \geq 0 \text{ a.e.}$$

which is equivalent to

$$h^2[E(w^2)\circ T^{-1}]^2 \leq M^2 h_2 E(w_2^2)\circ T^{-2} \text{ a.e.}$$

Corollary 2.4 — W is M-quasihyponormal if and only if W is M-paranormal.

Theorem 2.5 — Suppose

(i) $(W^* W)^K \leq W^{*K} W^K$,

(ii) $(W^* W)^{K-1} \geq W^{*K-1} W^{K-1}$ and

(iii) $T^{-1}\Sigma = \Sigma$

Then W is hyponormal.

PROOF : (i) and (iii) $\Rightarrow h_K [w_K^2 \circ T^{-K}] \geq J^K$

where $$J = hw^2 \circ T^{-1} \quad \text{... (A)}$$

(ii) and (iii) $\Rightarrow J^{K-1} \geq h_{K-1} w_{K-1}^2 \circ T^{-K+1}$. ... (B)

Also

$$w_K = w_{K-1}(w \circ T^{K-1})$$

and so

$$w_K^2 \circ T^{-K} = [w_{K-1}^2 \circ T^{-K}][(w\circ T^{K-1})^2 \circ T^{-K}].$$

Use this in (A) to obtain

$$(h_K \circ T)[w_{K-1}^2 \circ T^{-K+1}]\, w^2 \geq J^K \circ T. \quad \text{... (C)}$$

Combining (B) and (C)

$$J^{K-1}(h_K \circ T)[w_{K-1}^2 \circ T^{-K+1}]\, w^2 \geq (J^K \circ T)\, h_{K-1}[w_{K-1}^2 \circ T^{-K+1}]$$

$$\Rightarrow J^{K-1}(h_K \circ T) w^2 \geq (J\circ T)^K h_{K-1}.$$

By Lemma 2.4 of Panayappan[4], $h_K \circ T = (h\circ T)\, h_{K-1}$ and so

$$J^{K-1}(h\circ T)\, w^2 \geq (J \circ T)^K$$

which means $$J^{K-1} \geq (J\circ T)^{K-1}.$$

Thus $J \geq J\circ T$ which is equivalent with W is hyponormal[3].

Theorem 2.6 — Let W be a partial isometry. If W is of class (M, k) then it is an isometry on L^2 (Supp W).

PROOF : W is a partial isometry

$$\Rightarrow W = WW^* W$$

$$\Rightarrow w(f\circ T) = w(h\circ T)\, E(w^2) f\circ T \text{ for every } f \in L^2$$

$$\Rightarrow (h\circ T)\, E(w^2) = 1 \text{ on the supp } w.$$

Since every partial isometry of (M, k) class is quasinormal, we have

$(h \circ T)\, E(w^2) = hE(w^2) \circ T^{-1}$ on the supp W. Thus, on the supp w, $hE(w^2) \circ T^{-1} = 1$. Hence

$$\| Wf \|^2 = \int_X h(E(w^2) \circ T^{-1})\, |f|^2\, d\lambda = \int_X |f|^2\, d\lambda$$

$$= \|f\|^2.$$

Thus W is an isometry.

Theorem 2.7 — Suppose $T^{-1}\Sigma = \Sigma$. Then W^* is of (M, k) class if and only if

$$w_K^2 (h_K \circ T^K) \geq w^{2K} (h \circ T)^K \text{ a.e.}$$

PROOF : Since $T^{-1}\Sigma = \Sigma$, E is an identity and so

$$W^{*K} f = h_K (w_K f) \circ T^{-K}.$$

Then W^* is of (M, k) class

$$\Leftrightarrow W^K W^{*K} \geq (WW^*)^K$$

$$\Leftrightarrow \langle (W^K W^{*K} - (WW^*)^K f, f \rangle \geq 0 \text{ for all } f \in L^2$$

$$\Leftrightarrow \int_X [w_K^2 (h_K \circ T^K) - w^{2K} (h \circ T)^K]\, |f|^2\, d\lambda \geq 0$$

$$\Leftrightarrow w_K^2 h_K \circ T^K \geq w^{2K} (h \circ T)^K \text{ a.e.}$$

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