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MEASURES IN BITOPOLOGICAL SPACES

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In this paper we attempt to develop an initial theory of outer measure and regularity of measure in a bitopological space. For this purpose, we need to define pairwise Borel sets and some allied classes of sets which are extensively required for our investigations.

1. INTRODUCTION

Recently Polexe[7] has introduced the notion of outer measure in a bitopological space $(X, \mathcal{P}, \mathcal{Q})$ where the topologies $\mathcal{P}$ and $\mathcal{Q}$ are generated by quasi pseudo metrics p and q respectively. The method of Polexe appears to be restrictional in the sense that his approach is not applicable to a general bitopological space. We observe here that following the traditional method and using certain specified classes of sets, one may generate an outer measure in a general bitopological space.

Integrations in a locally compact space and its developments depend essentially on the regularity of measures. In a bitopological settings if one wishes to study the theory of integrations, the foremost necessity is the consideration of (pairwise) regularity of measures which we attempt to highlight in the last section. But before this, we need some set theoretic analysis which we do in section 3.

Throughout $(X, \mathcal{P}, \mathcal{Q})$ stands for a bitopological space and R for the real number space, unless otherwise stated.

2. OUTER MEASURE IN BITOPOLOGICAL SPACES

Definition 1 (Kelly[5]) — A set X on which are defined two arbitrary topologies $\mathcal{P}, \mathcal{Q}$ is called a bitopological space and is denoted by $(X, \mathcal{P}, \mathcal{Q})$.

Definition 2 (Fletcher *et al.*[3]) — A cover $\mathcal{B}$ of $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise open if $\mathcal{B} \subset \mathcal{P} \cup \mathcal{Q}$ and both $\mathcal{B} \cap \mathcal{P}$ and $\mathcal{B} \cap \mathcal{Q}$ contain nonempty sets.

Definition 3 — Let p be a pseudometric (metric) on a nonempty set X. The diameter of $A \subset X$ denoted by $d_p(A)$, is defined by $d_p(A) = \sup \{p(x, y); x, y \in A\}$.

We outline below the method of constructing an outer measure in $(X, \mathcal{P}, Q)$ as given in Polexe[7] where, however, no proof is given.

Let $(X, \mathcal{P}, Q)$ be a bitopological space where $\mathcal{P}$ and Q are the topologies induced respectively by a quasi pseudo metric p and its conjugate quasi pseudo metric q. Let $\sigma : \mathcal{P}(X) \to R$ be a non-negative set function with $\sigma(\phi) = 0$. For arbitrary

$A \subset X, n \in N$, define

$$\mu_p^n(A) = \inf \left(\sum_i \sigma(A_i) \right)$$

where the infimum runs over all countable collection of subsets $\{A_i\}$ of X with $A \subset \bigcup A_i$ and $d_p(A_i) \leq 1/n$. Then clearly $\mu_p^n(A) \leq \mu_p^{n+1}(A)$. Let

$$\mu_p(A) = \sup_n \mu_p^n(A).$$

Finally let us define

$$\mu(A) = \max(\mu_p(A), \mu_q(A)).$$

Then μ is an outer measure on the hereditary σ-ring $\mathcal{P}(X)$.

Although the proof of the above is almost routine, for the sake of completeness we outline the proof.

PROOF : Since σ is non-negative, so also is μ. Again since $\sigma(\phi) = 0, \mu(\phi) = 0$.

Let $B \subset C$. Then any countable collection of subsets $\{A_i\}$ of X which covers C also covers B. Hence $\mu_p^n(B) \leq \mu_p^n(C)$, for all $n \in N$ and so $\mu_p(B) \leq \mu_p(C)$. Similarly $\mu_q(B) \leq \mu_q(C)$. Therefore $\mu(B) \leq \mu(C)$.

Let $B = \bigcup_{k=1}^{\infty} B_k$. Let $\varepsilon > 0$ be arbitrary. There is a countable collection of subsets $\{A_{ki}\}_{i \in \Delta_k}$ of X which covers B_k, $d_p(A_{ki}) \leq 1/n$ such that

$$\mu_p^n(B_k) + \varepsilon/2^k > \sum_{i \in \Delta_k} \sigma(A_{ki}).$$

Clearly $\{A_{ki}; i \in \Delta_k, k = 1, 2, \ldots\}$ is a countable collection of subsets of X which covers B. So

$$\mu_p^n(B) \leq \sum_{k=1}^{\infty} \left(\sum_{i \in \Delta_k} \sigma(A_{ki}) \right) \leq \sum_{k=1}^{\infty} (\mu_p^n(B_k) + \varepsilon/2^k) \leq \sum_{k=1}^{\infty} \mu_p^n(B_k) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mu_p^n(B) \leq \sum_{k=1}^{\infty} \mu_p^n(B_k)$, so $\mu_p(B) \leq \sum_{k=1}^{\infty} \mu_p(B_k)$. Similarly $\mu_q(B) \leq \sum_{k=1}^{\infty} \mu_q(B_k)$ and hence $\mu(B) \leq \sum_{k=1}^{\infty} \mu(B_k)$. This completes the proof.

It is evident that the above construction as given in Polexe[7] depends on the quasi pseudo metrics p and q and so is not general. We outline below a method of constructing an outer measure applicable to a general bitopological space.

Let $(X, \mathcal{P}, \mathcal{Q})$ be a bitopological space. Let $\sigma : (\mathcal{P} \cup \mathcal{Q}) \to R$ be a non-negative set function with $\sigma(\phi) = 0$. Define for arbitrary $A \subset X$

$$\nu(A) = \inf \left(\sum_i \sigma(A_i) \right)$$

where the infimum runs over all countable pairwise open cover $\{A_i\}$ of A. Then ν is an outer measure on $\mathcal{P}(X)$.

The proof is omitted as being similar to a (single) topological space.

Note 1 : The set of all ν-measurable sets in $(X, \mathcal{P}, \mathcal{Q})$ is obtained in the usual way, which form a σ-ring on which the restriction of ν is a measure.

We give examples of outer measures in $(X, \mathcal{P}, \mathcal{Q})$ obtained by the general procedure.

Example 1 — Let $X = R$, and $\mathcal{P}$ and $\mathcal{Q}$ respectively be the lower limit topology and upper limit topology on R. Then $(X, \mathcal{P}, \mathcal{Q})$ is a bitopological space. Let the set function σ be the Lebesgue outer measure η(say) on R. Let ν be the outer measure on $(X, \mathcal{P}, \mathcal{Q})$. Let $A \subset X$. If $\{A_i\}$ be a countable collection of open intervals with $A \subset \bigcup_i A_i$, then $\nu(A) \leq \sum_i \eta(A_i)$. Thus $\nu(A) \leq \inf \{ \sum_i \eta(A_i);\ \{A_i\}$ runs over all countable collection of open intervals with $A \subset \bigcup_i A_i \} = \eta(A)$. If $\eta(A) > \nu(A)$, there is a countable pairwise open cover $\{A_i\}$ of A such that $\sum_i \eta(A_i) < \eta(A)$. Here A_i is semiclosed. Let $0 < \varepsilon < \eta(A) - \sum_i \eta(A_i)$. If $A_i = [a_i, b_i)$ or $(a_i, b_i]$ let $B_i = (a_i - \varepsilon/2^{i+1}, b_i + \varepsilon/2^{i+1})$, then $A \subset \bigcup_i B_i$, B_i is an open interval and $\sum_i \eta(B_i) \leq \sum_i \eta(A_i) + \varepsilon < \eta(A)$, a contradiction. Hence $\nu(A) = \eta(A)$. Thus here ν is the Lebesgue outer measure on R.

Example 2 — Let $X = R$. Let $\mathcal{P}$ be the usual topology on R. Let $\mathcal{Q} = \{\phi\} \cup \{G_\alpha\}$ where G_α are of the form $G_\alpha = U \cup (1, 2)$, $U \in \mathcal{P}$. Then $(X, \mathcal{P}, \mathcal{Q})$ is a bitopological space. We take the Lebesgue outer measure η(say) for the set function

σ. Let ν be the outer measure on $(X, \mathcal{P}, \mathcal{Q})$. Let $A = (3, 5)$. Then $\eta(A) = 2$. Now since $(3, 5) \cup (1, 2)$ is a countable pairwise open cover of A, $\nu(A) \leq \eta(3, 5) + \eta(1, 2) = 3$. If $\nu(A) < 3$ then for k with $\nu(A) < k < 3$, there is a countable pairwise open cover $\{A_i\}$ of A such that $\sum_i \eta(A_i) < k$. But since $\bigcup_i A_i \supset A \cup (1, 2)$, we have $\sum_i \eta(A_i) \geq \eta[A \cup (1, 2)] = 3 > k$, a contradiction. Hence $\nu(A) = 3$. Thus ν is different from η.

3. Pairwise Borel Sets

The following definitions and analysis are needed to introduce the idea of pairwise Borel sets in $(X, \mathcal{P}, \mathcal{Q})$ which will be of use in future developments.

Definition 4 (Kelly[5]) — $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise Hausdorff if for any two distinct points x, y of X, there exists $U \in \mathcal{P}, V \in \mathcal{Q}$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

Definition 5 (Fletcher *et al.*[3]) — $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise compact if every pairwise open cover of X has a finite subcover.

Remark 1 : In a Hausdorff (single) topological space compact subsets are always closed, but in a pairwise Hausdorff bitopological space $(X, \mathcal{P}, \mathcal{Q})$, pairwise compact sets may not be $\mathcal{P}$ or $\mathcal{Q}$ closed as shown by the following example.

Example 3 — Let $X = [0, \infty)$. Let $\mathcal{U}$ denote the class of all open sets in the usual topology on X. Let $\mathcal{P} = \{\phi\} \cup \{G\}$ where G is of the form $G = (0, x) \cup K$ where $x > 0$, $K \in \mathcal{U}$. Let $\mathcal{Q} = \{\phi\} \cup \{G_1\}$ where G_1 is of the form $G_1 = (x, \infty) \cup K$ where K denotes all sets $U \in \mathcal{U}$ and the sets $\{0\} \cup U$, $U \in \mathcal{U}$. Then $(X, \mathcal{P}, \mathcal{Q})$ is a bitopological space. Let $x, y \in X, x \neq y\ (y \neq 0)$. If $x < y$ then $x \in [0, x_1) \in \mathcal{P}$, $y \in (x_1, \infty) \in \mathcal{Q}$ and $[0, x_1) \cap (x_1, \infty) = \phi$, where $x < x_1 < y$. Again if $x > y$, choosing $0 < y_1 < y < y_2 < x < y_3$ we get $x \in (0, y_1) \cup (y_2, y_3) \in \mathcal{P}$, $y \in (y_1, y_2) \cup (y_3, \infty) \in \mathcal{Q}$ and $[(0, y_1) \cup (y_2, y_3)] \cap [(y_1, y_2) \cup (y_3, \infty)] = \phi$. Finally if $y = 0$ then $x \in (0, x_1) \in \mathcal{P}$, $y \in (\{0\} \cup (x_1, \infty)) \in \mathcal{Q}$ and $(0, x_1) \cap [\{0\} \cup (x_1, \infty)] = \phi$ where $0 < x < x_1$. Thus $(X, \mathcal{P}, \mathcal{Q})$ is a pairwise Hausdorff space.

Now let $A = (0, \infty)$. Let $\mathcal{B}$ be a pairwise open cover of A. Then $\mathcal{B}$ contains at least one $V \in \mathcal{P}, W \in \mathcal{Q}$, $V \neq \phi \neq W$. So there are v, $w > 0$ such that $(0, v) \subset V$, $(w, \infty) \subset W$. Now $\mathcal{B}$ is a pairwise open cover of $[v, w]$. Since the members of $A \cap \mathcal{B} = \{A \cap U; U \in \mathcal{B}\}$ are open sets of the usual topology which also covers $[v, w]$, there is a finite subcollection of $A \cap \mathcal{B}$ which covers $[v, w]$. Then the corresponding members of $\mathcal{B}$ together with V, W form a finite subcover of A. Hence A is pairwise compact.

But it is easy to verify that A is neither $\mathcal{P}$ closed nor Q closed.

As such in contrast with a (single) topological space, in the following definition we need the additional assumption on the set being $\mathcal{P}$-closed or Q-closed and the assumption of being pairwise Hausdorff do not serve the purpose and so omitted.

Definition 6 — In $(X, \mathcal{P}, Q)$, the σ-ring generated by the collection of all pairwise compact sets which are either $\mathcal{P}$-closed or Q-closed is called the class of pairwise Borel sets.

Definition 7 — A set A in $(X, \mathcal{P}, Q)$ is said to be bounded if it is contained in a pairwise compact set. A is called σ-bounded if it is contained in the union of a sequence of pairwise compact sets.

Theorem 1 — The pairwise Borel sets are σ-bounded.

The proof is omitted.

Definition 8 — In $(X, \mathcal{P}, Q)$ the σ-ring generated by all $\mathcal{P}$ and Q closed sets is called the class of all pairwise weakly Borel sets.

Theorem 2 — The class of pairwise Borel sets is contained in the class of all σ-bounded pairwise weakly Borel sets.

The proof is omitted.

Remark 2 : Unlike a (single) topological space (Berberian[1], p.181) (assumed to be Hausdorff), the converse of Theorem 2 is not true in $(X, \mathcal{P}, Q)$ as shown by the following example.

Example 4 — Let $X = R$ and $\mathcal{P} = \{\phi, R, R^-, R^- \cup A_\alpha\}$ where R^- is the set of negative real numbers and A_α runs over all subsets of R^+, the set of all non-negative real numbers. Let $Q = (\phi, R, B_\alpha\}$ where B_α runs over all subsets of R^-. Then $(X, \mathcal{P}, Q)$ is a bitopological space. Now $R - \{-1\}$ is Q-closed, $R^+ - \{1\}$ is $\mathcal{P}$-closed. Then $[R - \{-1\}] - [R^+ - \{1\})] = (R^- - \{-1\}) \cup \{1\} = A$(say) is a pairwise weakly Borel set which is also σ-bounded, since R^- and $\{1\}$ are pairwise compact.

We observe that the only $\mathcal{P}$-closed sets are R, ϕ and all sets of the form A_α and the only Q-closed sets are R, ϕ and all sets of the form $R^+ \cup B_\alpha$ of which only the sets A_α are pairwise compact. Thus all the pairwise Borel sets are subsets of R^+. Therefore A is not a pairwise Borel set.

4. Pairwise Regular Borel Measure

Definition 9 — A measure μ defined on the class of all pairwise Borel sets in $(X, \mathcal{P}, Q)$ with $\mu(C) < \infty$ for each pairwise compact member C is called a pairwise Borel measure.

Definition 10 (Reilly[8]) — In $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is called locally compact with respect to $\mathcal{Q}$ if each point of X has a $\mathcal{P}$-open neighbourhood whose $\mathcal{Q}$-closure is pairwise compact. $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise locally compact if both $\mathcal{P}$ and $\mathcal{Q}$ are locally compact with respect to each other.

We like to define now pairwise regularity of Borel measure. We assume $(X, \mathcal{P}, \mathcal{Q})$ to be pairwise locally compact. We require the following classes of subsets of $(X, \mathcal{P}, \mathcal{Q})$ which will be used in our developments.

Let $\mathcal{V}^*$ = The class of all subsets of X which can be expressed as the union of a countable number of sets of the form $P \cap Q, P \in \mathcal{P}, Q \in \mathcal{Q}$.

$\mathcal{T}$ = The class of all pairwise Borel sets and μ be a pairwise Borel measure.

C = The subfamily of $\mathcal{T}$ whose members can be expressed as the intersection of a countable number of sets of the form $C_1 \cup C_2$, C_1 and C_2 are pairwise compact which are $\mathcal{P}$ or $\mathcal{Q}$ closed.

$\mathcal{V}$ = The subfamily of $\mathcal{T}$ whose members are also members of $\mathcal{V}^*$.

Note 3 : When $\mathcal{P} = \mathcal{Q}$ and $(X, \mathcal{P})$ is Hausdorff, $\mathcal{T}$ becomes the class of Borel sets. Also since all sets $P \cap Q, P \in \mathcal{P}, Q \in \mathcal{Q}$ become open, $\mathcal{V}^*$ becomes the class of all open sets and so $\mathcal{V}$ coincides with the class of all open Borel sets. Further C becomes the class of all compact sets.

We can now introduce the definition of pairwise regularity as follows :

A set $A \in \mathcal{T}$ is called pairwise outer regular if

$$\mu(A) = \inf\{\mu(U); A \subset U \in \mathcal{V}\}.$$

A set $A \in \mathcal{T}$ is called pairwise inner regular if

$$\mu(A) = \sup\{\mu(C); A \supset C \in C\}.$$

where the infimum and supremum exist by Lemma 4 and Lemma 3 given below. A set $A \in \mathcal{T}$ is called pairwise regular if it is both pairwise outer regular and pairwise inner regular. If every $A \in \mathcal{T}$ is pairwise regular then μ is called a pairwise regular Borel measure on $\mathcal{T}$.

In a locally compact topological space, the axioms (Berberian[1], p. 186) needed prior to the introduction of regularity of Borel measure automatically holds in respect of compact sets, Borel sets, open Borel sets and Borel measure. These axioms have been frequently used to establish the basic properties of regular Borel measure. But in a bitopological space, the matter does not appear to be analogous. Here we need to establish the contents of the axioms for specified classes of sets which we do through the Lemmas those follow and these will be required extensively to obtain the basic properties of a pairwise Borel measure in respect of pairwise regularity.

Lemma 1 — For any pairwise compact set C, there is a bounded $U \in \mathcal{V}^*$ such that $C \subset U$.

PROOF : Since $(X, \mathcal{P}, \mathcal{Q})$ is pairwise locally compact, for each $x \in C$, there exist $P_x \in \mathcal{P}$ and $Q_x \in \mathcal{Q}$ such that $x \in P_x$, $x \in Q_x$ and $\mathcal{Q}\text{-}cl(P_x)$ and $\mathcal{P}\text{-}cl(Q_x)$ are pairwise compact. Then $\mathcal{B} = \{P_x, Q_x;\ x \in C\}$ form a pairwise open cover of C. Since C is pairwise compact, there are $\{U_1, ..., U_n\} \subset \mathcal{B}$ such that $C \subset \bigcup_{i=1}^{n} U_i = U$ (say). Now $U = \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (U_i \cap X) \in \mathcal{V}^*$. Also $U \subset \bigcup_{i=1}^{n} (\tau\text{-}cl(U_i))$ where $\tau = \mathcal{P}$ or $\mathcal{Q}$ according as $U_i \in \mathcal{Q}$ or $U_i \in \mathcal{P}$, which being a finite union of pairwise compact sets is also pairwise compact. Hence U is bounded.

Note 4 : In Lemma 1 U is of the form $P \cup Q$, $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. We can easily construct U in such a way that $P \cap C \neq \phi \neq Q \cap C$. This fact will be needed in Lemma 2.

Definition 11 (Fletcher[2], Lane[6]) — In $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is said to be completely regular with respect to $\mathcal{Q}$ if for each $\mathcal{P}$-closed set C and each point $x \notin C$, there is a real valued function f on X onto $[0, 1]$ such that $f(x) = 0$, $f(C) = 1$ and f is $\mathcal{P}$-upper semi continuous and $\mathcal{Q}$-lower semi continuous. $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise completely regular if both $\mathcal{P}$ and $\mathcal{Q}$ are completely regular with respect to each other.

We shall also frequently make use of the following theorem.

Theorem 3 (Reilly[8]) — If $(X, \mathcal{P}, \mathcal{Q})$ is pairwise compact and A is a subset of X which is $\mathcal{P}$ or $\mathcal{Q}$ closed then A is pairwise compact.

From this stage onwards, we assume $(X, \mathcal{P}, \mathcal{Q})$ to be pairwise completely regular also.

Lemma 2 — For any pairwise compact set C, there is a $U \in \mathcal{V}$ such that $C \subset U$. Further U can be chosen to be bounded.

PROOF : By Lemma 1 (with Note 4) we can find a bounded $V \in \mathcal{V}^*$ such that $C \subset V$, $V = P \cup Q, P \in \mathcal{P}$, $Q \in \mathcal{Q}$ and $P \cap C \neq \phi \neq Q \cap C$. Let $x \in C$. If $x \in P$, there exists a function f_x on X such that $f_x(x) = 0$, $f_x(y) = 1$ for all $y \in X - P$, $0 \leq f_x \leq 1$ and f_x is $\mathcal{P}$-upper semi continuous and $\mathcal{Q}$-lower semi continuous. If $x \in Q$, we obtain a similar function. Since $P \cap C \neq \phi \neq Q \cap C$, it follows that the collection of the sets $U_x = \{y; f_x(y) < 1/2\}$ when x varies over C, is formed with both $\mathcal{P}$ and $\mathcal{Q}$ open sets and so form a pairwise open cover of C. If x belongs to both $\mathcal{P}$ and $\mathcal{Q}$ then the modification is evident. There are $U_{x_1}, ..., U_{x_n}$ such that $C \subset \bigcup_{i=1}^{n} U_{x_i}$. Let $g = \min\ \{f_{x_1}, ..., f_{x_n}\}$. Then $g(y) = 1$ for all $y \in X - (P \cup Q) = X - V$, $g(x) < 1/2$ for all $x \in C$. Then $C \subset \{x;\ g(x) < 1/2\} = U$ (say) $\subset V$. Now

$U = \{x;\ g(x) < 1/2\} = \bigcup_{i=1}^{n} \left\{x\,; f_{x_i}(x) < 1/2\right\} \in \mathcal{V}^*$. Also $U = \bigcup_{k=2}^{\infty} \left\{x\,; g(x) \le 1/2 - 1/2^k\right\}$. For each k, $\{x\,; g(x) \le 1/2 - 1/2^k\} = \bigcup_{i=1}^{n} \left\{x\,; f_{x_i}(x) \le 1/2 - 1/2^k\right\}$. Since $\{x\,; f_{x_i}(x) \le 1/2 - 1/2^k\}$ is $\mathcal{P}$ or Q closed and $\{x\,; f_{x_i}(x) \le 1/2 - 1/2^k\} \subset \{x\,; g(x) \le 1/2 - 1/2^k\} \subset \{x\,; g(x) < 1/2\} \subset V$ where V is bounded, by Theorem 3, $\{x\,; f_{x_i}(x) \le 1/2 - 1/2^k\}$ is pairwise compact. So $\{x\,; g(x) \le 1/2 - 1/2^k\}$ is a pairwise Borel set for each k. Then U is also pairwise Borel and so $U \in \mathcal{V}$. Also since $U \subset V$, U is bounded.

Lemma 3 — C is closed under finite union, countable intersection. Also $\phi \in C$ and $\mu(C) < \infty \;\; \forall \;\; C \in C$.

PROOF : The proof is straightforward and is omitted.

Lemma 4 — $\mathcal{V}$ is closed under finite intersection and countable union. Also for each $E \in \mathcal{T}$ there is a $\mathcal{U} \in \mathcal{V}$ such that $E \subset U$.

PROOF : The first part is clear. By Theorem 1, E is σ-bounded i.e. $E \subset \bigcup_{k=1}^{\infty} C_k$ where C_k is pairwise compact. By Lemma 2, there is a $U_k \in \mathcal{V}$ such that $C_k \subset U_k$. Then $E \subset \bigcup_{k=1}^{\infty} U_k = U$ (say) $\in \mathcal{V}$.

Lemma 5 — If $U \in \mathcal{V}$, $C \in C$ then $U - C \in \mathcal{V}$.

Lemma 6 — If $C \in C$, then there is a bounded $U \in \mathcal{V}$ such that $C \subset U$.

Lemma 7 — $\mathcal{T}$ is the σ-ring generated by C.

The proofs of the above Lemmas are omitted.

After the above Lemmas are obtained, the proofs of some of the theorems on pairwise regularity of Borel measure become analogous to the corresponding proofs in (single) topological spaces (Berberian[1], p. 186). We only state these theorems, some of which will be used below to obtain other basic properties.

Theorem 4 — Every set in $\mathcal{V}$ is pairwise outer regular and every set in C is pairwise inner regular.

Theorem 5 — The union of a sequence of pairwise outer regular (inner regular) sets is pairwise outer regular (inner regular).

Lemma 4 is required for the proof.

Theorem 6 — The intersection of a sequence of pairwise outer (inner) regular sets of finite measures is pairwise outer (inner) regular.

Lemma 3 is required for the proof.

Theorem 7 — If every set in C is pairwise outer regular then $C - D$ is pairwise outer regular $\forall$ $C, D \in C$. If every bounded set in $\mathcal{V}$ is pairwise inner regular then so is each set $C - D$ $\forall$ $C, D \in C$.

Lemmas 3, 5 and 6 are required for the proof.

Theorem 8 — If every bounded set in $\mathcal{V}$ is pairwise inner regular then every set in C is pairwise outer regular.

Lemmas 5 and 6 are required for the proof.

Theorem 9 — The necessary and sufficient condition for μ to be pairwise regular on $\mathcal{T}$ is that every bounded set in $\mathcal{V}$ is pairwise inner regular.

Lemma 7 and Theorems 5, 6, 7 and 8 are required for the proof.

Theorem 10 — If $\mathcal{T}$ satisfies the condition.

(*) : For each bounded $U \in \mathcal{V}$, there is a $D \in C$ and a set C which is both $\mathcal{P}$ and $\mathcal{Q}$ closed such that $U \subset C \subset D$; then the pairwise outer regularity of all sets of $\mathcal{T}$ of the form $A - B$, $A \in C$, $B \in \mathcal{V}$, $B \subset A$ implies the pairwise regularity of μ on $\mathcal{T}$.

PROOF : We shall only show that each bounded set in $\mathcal{V}$ is pairwise inner regular. The theorem then follows from Theorem 9. Let U be a bounded set in $\mathcal{V}$. By (*), there is a $D \in C$ such that $U \subset C \subset D$ where C is both $\mathcal{P}$ and $\mathcal{Q}$ closed. Now $D - U$ is pairwise outer regular. So given $\varepsilon > 0$, there is a $G \in \mathcal{V}$ such that $D - U \subset G$ and $\mu(G) - \mu(D - U) < \varepsilon$. Let $K = C - G$. Since $D \in C$, D is bounded i.e. $D \subset C_1$ where C_1 is pairwise compact and so C is pairwise compact (see Theorem 3). Now $G = \bigcup_{i \in \Delta} (P_i \cap Q_i)$ where $P_i \in \mathcal{P}$, $Q_i \in \mathcal{Q}$ and Δ is countable. Then

$$\begin{aligned} C - G &= C \cap G' \\ &= C \cap \left[\bigcap_{i \in \Delta} (P_i' \cup Q_i') \right] \\ &= \bigcap_{i \in \Delta} \left[(P_i' \cap C) \cup (Q_i' \cap C) \right] \in C \end{aligned}$$

since $C \cap P_i'$ and $C \cap Q_i'$ are pairwise compact and $\mathcal{P}$ and $\mathcal{Q}$ closed respectively (where dash denotes the complement).

Now $K = C - G \supset U - G$. Again $x \in C - G \Rightarrow x \in C$ but $x \notin G \Rightarrow x \in U$ and $x \notin G$ [since $x \notin U \Rightarrow x \in C - U \subset D - U \subset G$] $\Rightarrow x \in U - G$. Thus $K = C - G = U - G \subset U$. Now $U - K = U - \bullet(U - G) = U \cap G \subset G - (D - U)$ [since $x \in U \cap G \Rightarrow x \in G$ and $x \in U \subset D \Rightarrow x \in G$ but $x \notin D - U$]. Then

$$\mu(U) - \mu(K) = \mu(U - K) \leq \mu(G - (D - U)) \leq \mu(G) - \mu(D - U) < \varepsilon.$$

Hence U is pairwise inner regular. This proves the theorem.

Let C^* denote the class of all pairwise compact sets that are $\mathcal{P}$ or $\mathcal{Q}$ closed. We show that under certain condition the pairwise inner regularity of μ is derived from the members of C^* only.

Theorem 11 — If for each $A \in \mathcal{T}$, $C \in C$, $C \subset A \Rightarrow C \subset K \subset A$ where $K \in C^*$ then

(a) μ is pairwise inner regular if and only if

(b) $\mu(A) = \sup \{\mu(K); A \supset K \in C^*\}$.

PROOF : Clearly (b) implies (a). Let (a) hold. If $A \in \mathcal{T}$ and $\varepsilon > 0$ be given, there is a $C \in C$ such that $C \subset A$, $\mu(A) - \mu(C) < \varepsilon$. By the given condition there is a $K \in C^*$ such that $C \subset K \subset A$. Then $\mu(A) - \mu(K) = \mu(A - K) \leq \mu(A - C) \leq \mu(A) - \mu(C) < \varepsilon$. Hence (b) holds.

Analogously we may obtain the following

Theorem 12 — If for each $A \in \mathcal{T}$, $U \in \mathcal{V}$, $A \subset U \Rightarrow A \subset G \subset U$ where $G \in \mathcal{T}$ is $\mathcal{P}$ or $\mathcal{Q}$ open then

(a) μ is pairwise outer regular if and only if

(b) $\mu(A) = \inf \{\mu(G)\,;\, A \subset G$ where $G \in \mathcal{T}$ is $\mathcal{P}$ or $\mathcal{Q}$ open$\}$.

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