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A SEMINUMERICAL ITERATIVE METHOD FOR SOLVING A MACHINE INTERFERENCE PROBLEM

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A seminumerical iterative method for solving a machine interference problem involving M machines and two repairmen each specializing in one type of stoppage with two kinds of stoppages in all is presented. Sufficient convergence conditions for the method are also discussed.

1. INTRODUCTION

Saaty[2] has discussed a machine interference problem involving M machines and c repairmen each specializing in one type of stoppage with c kinds of stoppages in all and has provided a solution of this problem. Here an alternative seminumerical iterative method for solving the machine interference problem involving M machines and two repairmen each specializing in one type of stoppage with two kinds of stoppages in all is presented. The alternative method is on the line of the seminumerical iterative method developed by Brandwajn[1] for solving the balance equations of finite two-dimensional birth and death processes. It uses the formal knowledge of the stationary probability distribution of one of the two state variables and the iteration is applied to the conditional probabilities of the second state variable given the probabilities of the first one. The alternative method is expected to perform better than that of Saaty[2] from the computational point of view. Sufficient convergence conditions for the method are also discussed. In the end, possible future extension of the present work is pointed out.

2. STATEMENT OF THE PROBLEM

Assume that the total number of machines is M. Let the stoppage distributions be Poisson for two different types of stoppages with corresponding parameters $\lambda_1 (n_1, n_2)$ and $\lambda_2 (n_1, n_2)$. Let the two repairmen be specialized in different types of stoppages. Let the repair distributions for the first and second type of stoppages be exponential with parameters $\mu_1 (n_1, n_2)$ and $\mu_2 (n_1, n_2)$ respectively. Let $p (n_1, n_2)$ be the steady-state probability that there are n_1 machines with stoppage

of type 1 and n_2 machines with that of type 2. Here n_1 and n_2 can take the values 0, 1,....., M subject to the condition $n_1+n_2 \leqslant M$. Then, we have

$$\lambda_1 (M, n_2) = 0 \ \forall \ n_2, \qquad \lambda_2 (n_1, M) = 0 \ \forall \ n_1,$$

$$\mu_1 (0, n_2) = 0 \ \forall \ n_2, \qquad \mu_2 (n_1, 0) = 0 \ \forall \ n_1.$$

The steady state equations are

$$\left\{\begin{array}{l} -[(M-n_1-n_2)\{\lambda_1(n_1,n_2)+\lambda_2(n_1,n_2)\}+\mu_1(n_1,n_2) \\ +\mu_2(n_1,n_2)]\,p(n_1,n_2)+(M-n_1-n_2+1)[\lambda_1(n_1-1,n_2) \\ \times p(n_1-1,n_2)+\lambda_2(n_1,n_2-1)\,p(n_1,n_2-1)] \\ +\mu_1(n_1+1,n_2)\ p(n_1+1,n_2)+\mu_2(n_1,n_2+1) \\ \times p(n_1,n_2+1)=0 \end{array}\right. \quad \ldots (1a)$$

$$n_1, n_2 = 1, \ldots\ldots M-1, n_1+n_2 \leqslant M-1$$

$$\left\{\begin{array}{l} -[\mu_1(n_1,n_2)\,p(n_1,n_2)+\mu_2(n_1,n_2)\,p(n_1,n_2)] \\ +\lambda_1(n_1-1,n_2)\,p(n_1-1,n_2)+\lambda_2(n_1,n_2-1) \\ \times p(n_1,n_2-1)=0 \end{array}\right. \quad \ldots (1b)$$

$$n_1, n_2 = 1, \ldots, M-1, n_1+n_2 = M$$

$$\left\{\begin{array}{l} -[(M-n_1)\{\lambda_1(n_1,0)+\lambda_2(n_1,0)\}+\mu_1(n_1,0)]\,p(n_1,0) \\ +(M-n_1+1)\lambda_1(n_1-1,0)\,p(n_1-1,0) \\ +\mu_1(n_1+1,0)\,p(n_1+1,0)+\mu_2(n_1,1)\,p(n_1,1)=0. \end{array}\right. \quad \ldots (1c)$$

$$\left\{\begin{array}{l} -[(M-n_2)\{\lambda_1(0,n_2)+\lambda_2(0,n_2)\}\,\mu_2(0,n_2)]\,p(0,n_2) \\ +(M-n_2+1)\lambda_2(0,n_2-1)\,p(0,n_2-1) \\ +\mu_2(0,n_2+1)\ p(0,n_2+1)+\mu_1(1,n_2)\ p(1,n_2)=0 \end{array}\right. \quad \ldots (1d)$$

$$\mu_1(M,0)\,p(M,0) = \lambda_1(M-1,0)\,p(M-1,0) \quad \ldots (1e)$$

$$\mu_2(0,M)\ p(0,M) = \lambda_2(0,M-1)\ p(0,M-1) \quad \ldots (1f)$$

$$M\{\mu_1(0,0)+\mu_2(0,0)\}\ p(0,0) = \mu_1(1,0)\ p(1,0) + \mu_2(0,1)\ p(0,1) \quad \ldots (1g)$$

Let us assume that such a stationary probability distribution exists for our model so that above equations have a unique solution which satisfies the normalization condition

$$\sum_{n_1=0}^{M} \ \sum_{n_2=0}^{M-n_1} p(n_1, n_2) = 1. \quad \ldots (2)$$

3. Iterative Method

It is easy to obtain the stationary probability distribution for one of the state variables n_1 and n_2, say n_1. Let the stationary probability distribution of n_1 be $p(n_1)$ and Let $p_{n_1}(n_2)$ be the stationary conditional probability of n_2 given n_1, i. e.

$$p_{n_1}(n_2) = \text{prob}\{n_2 \mid n_1\}.$$

Our model has the same probability distribution $p(n_1)$ for the steady state as a one-dimensional birth and death process with birth rate

$$l(n_1) = \sum_{n_2=0}^{M} p_{n_1}(n_2)\,\lambda_1(n_1, n_2), \quad n_1 = 0, \ldots M-1 \qquad \ldots(3)$$

and death rate

$$m(n_1) = \sum_{n_2=0}^{M} p_{n_1}(n_2)\,\mu_1(n_1, n_2), \quad n_1 = 1, \ldots, M \qquad \ldots(4)$$

$p(n_1)$ is then given by

$$p(n_1) = 1/G \prod_{i=1}^{n_1} l(i-1)/m(i), \quad n_1 = 0, \ldots M \qquad \ldots(5)$$

where an empty product is assigned the value $+1$, and G is a normalization constant.

Now

$$p(n_1, n_2) = p(n_1)\,p_{n_1}(n_2). \qquad \ldots(6)$$

Using equations (3), (4), (5), and (6), the steady state equations for our model can be written in terms of conditional probabilities as follows

$$\begin{cases} -[(M-n_1-n_2)\{\lambda_1(n_1, n_2)+\lambda_2(n_1, n_2)\}+\mu_1(n_1, n_2) \\ +\mu_2(n_1, n_2)]\,p_{n_1}(n_2)+(M-n_1-n_2+1)\,\lambda_2(n_1, n_2-1) \\ \times p_{n_1}(n_2-1)+\mu_2(n_1, n_2+1)\,p_{n_1}(n_2+1) \\ +(M-n_1-n_2+1)\,m(n_1)\,\lambda_1(n_1-1, n_2)\,p_{n_1-1}(n_2)/l(n_1-1) \\ +l(n_1)\,\mu_1(n_1+1, n_2). \\ p_{n_1+1}(n_2)/m(n_1+1)=0 \end{cases} \qquad \ldots(7a)$$

$$n_1, n_2 = 1, \ldots, M-1, \quad n_1 + n_2 \leqslant M-1$$

$$\left\{\begin{array}{l} -[\mu_1(n_1, n_2) + \mu_2(n_1, n_2)]\ p_{n_1}(n_2) + m(n_1)\ \lambda_1(n_1 - 1, n_2) \\ \times p_{n_1 - 1}(n_2)/l(n_1 - 1) + \lambda_2(n_1, n_2 - 1)\ p_{n_1}(n_2 - 1) = 0 \end{array}\right. \quad \ldots(7b)$$

$$n_1, n_2 = 1, \ldots\ldots, M-1, \quad n_1 + n_2 = M.$$

$$\left\{\begin{array}{l} -[(M - n_1)\ \{\lambda_1(n_1, 0) + \lambda_2(n_1, 0)\} + \mu_1(n_1, 0)]\ p_{n_1}(0) \\ + (M - n_1 + 1)\ m(n_1)\ \lambda_1(n_1 - 1, 0)\ p_{n_1 - 1}(0)/l(n_1 - 1) \\ + l(n_1)\ \mu_1(n_1 + 1, 0)\ p_{n_1 + 1}(0)/m(n_1 + 1) \\ + \mu_2(n_1, 1)\ p_{n_1}(1) = 0 \end{array}\right. \quad \ldots(7c)$$

$$\left\{\begin{array}{l} -[(M - n_2)\ \{\lambda_1(0, n_2) + \lambda_2(0, n_2)\} + \mu_2(0, n_2)]\ p_0(n_2) \\ + (M - n_2 + 1)\ \lambda_2(0, n_2 - 1)\ p_0(n_2 - 1) \\ + \mu_2(0, n_2 + 1)\ p_0(n_2 + 1) + l(0)\ \mu_1(1, n_2)\ p_1(n_2)/m(1) = 0 \end{array}\right. \quad \ldots(7d)$$

$$\mu_1(M, 0)\, p_M(0) = m(M)\, \lambda_1(M - 1, 0)\, p_{M-1}/l(M - 1) \quad \ldots(7e)$$

$$\mu_2(0, M)\, p_0(M) = \lambda_2(0, M - 1)\, p_0(M - 1) \quad \ldots(7f)$$

$$M[\mu_1(0, 0) + \mu_2(0, 0)]\ p_0(0) = l(0)\ \mu_1(1, 0)\ p_1(0)/m(1) + \mu_2(0, 1)\ p_0(1). \quad \ldots(7g)$$

Equations (7) involve only the birth and death rates and the conditional probabilities $p_{n_1}(n_2)$. It follows from the assupmtion on the uniqueness of the solution of (1) that equations (7) have a unique solution satisfying the probability condition

$$\sum_{n_2=0}^{M} p_{n_1}(n_2) = 1 \quad \text{for all} \quad n_1 = 0, \ldots, M.$$

Now the steady state distribution $p(n_1, n_2)$ can be computed from eqn. (6) using (3), (4) and (5) provided $p_{n_1}(n_2)$, has been computed. To compute $p_{n_1}(n_2)$, we propose an iterative method for solving eqns. (7). Use a superscript to indicate the iteration number. Choose initial distribution $p^0_{n_1}(n_2)$ so that neither $m^0(n_1)$ nor $l^0(n_1)$ is zero. $m^i(n_1)$ and $l^i(n_1)$, $i = 0, 1, \ldots$, are given by eqns. (3) and (4) with $p^i_{n_1}(n_2)$ instead of $p_{n_1}(n_2)$.

Now we start from $n_1 = 0$ and find the successive iteratives. For the ith ($i = 1, 2, \ldots$) iteration, we have

$$\left\{\begin{array}{l} -\left[(M - n_1 - n_2)\ \{\lambda_1(n_1, n_2) + \lambda_2(n_1, n_2)\} + \mu_1(n_1, n_2)\right. \\ \left. + \mu_2(n_1, n_2)\right]\ p^i_{n_1}(n_2) + (M - n_1 - n_2 + 1)\ \lambda_2(n_1, n_2 - 1) \\ \quad p^i_{n_1}(n_2 - 1) + \mu_2(n_1, n_2 + 1)\ p^i_{n_1}(n_2 + 1) \\ + (M - n_1 - n_2 + 1)\ m^i(n_1)\ \lambda_1(n_1 - 1, n_2)\ p^i_{n_1 - 1}(n_2) / \\ \quad l^i(n_1 - 1) + l^i(n_1)\ \mu_1(n_1 + 1, n_2)\ p^{i-}_{n_1 + 1}(n_2)/ \\ \quad m^{i-1}(n_1 + 1) = 0 \end{array}\right. \qquad \ldots(8a)$$

$$n_1, n_2 = 1, \ldots, M - 1, \quad n_1 + n_2 \leqslant M - 1$$

$$\left\{\begin{array}{l} -\left[\mu_1(n_1, n_2) + \mu_2(n_1, n_2)\right]\ p^i_{n_1}(n_2) + m^i(n_1)\,\lambda_1(n_1 - 1, n_2) \\ \quad p^i_{n_1 - 1}(n_2) / l^i(n_1 - 1) + \lambda_2(n_1, n_2 - 1) \\ \quad p^i_{n_1}(n_2 - 1) = 0 \end{array}\right. \qquad \ldots(8b)$$

$$n_1, n_2 = 1, \ldots, M - 1, \quad n_1 + n_2 = M$$

$$\left\{\begin{array}{l} -(M - n_1)\ \{\lambda_1(n_1, 0) + \lambda_2(n_1, 0)\} + \mu_1(n_1, 0)\ ]\ p^i_{n_1}(0) \\ + (M - n_1 + 1)\ m^i(n_1)\ \lambda_1(n_1 - 1, 0)\ p^i_{n_1 - 1}(0)/l^i(n_1 - 1) \\ + l^i(n_1)\ \mu_1(n_1 + 1, 0)\ p^{i-1}_{n1 + 1}(0)/m^{i-1}(n_1 + 1) \\ + \mu_2(n_1, 1)\ p_{n_1^i}(1) = 0. \end{array}\right. \qquad \ldots(8c)$$

$$\left\{\begin{array}{l} -\left[(M - n_2)\ \{\lambda_1(0, n_2) + \lambda_2(0, n_2)\} + \mu_2(0, n_2)\right]\ p^i_0(n_2) \\ + (M - n_2 + 1)\ \lambda_2(0, n_2 - 1)\ p^i_0(n_2 - 1) \\ + \mu_2(0, n_2 + 1) + p^i_0(n_2 + 1) + l^i(0)\ \mu_1(1, n_2)\ p^{i-1}_1(n_2) = 0 \end{array}\right. \qquad \ldots(8d)$$

$$\mu_1(M,0)\ p^i_M(0) = m^i(M)\ \lambda_1(M - 1,0)\, p^i_{M-1}(0)/l^i(M - 1) \qquad \ldots(8e)$$

$$\mu_2(0, M)\, p_0^i(M) = \lambda_2(0, M-1)\; p_0^i(M-1) \qquad \ldots(8f)$$

$$M[\,\mu_1(0,0) + \mu_2(0,0)\,]\; p_0^i(0)$$
$$= l^i(0)\;\; \mu_1(1,0)\;\; p_1^{i-1}(0)/\, m^{i-1}(1) + \mu_2(0,1)\;\; p_0^i(1) \qquad \ldots(8g).$$

$$\sum_{n_2=0}^{M} p_{n_1}^i(n_2) = 1 \qquad \ldots(9)$$

for consecutive values of n_1.

Note that for each n_1, eqns. (8) constitute a relatively simple recurrence relation for the corresponding $p_{n_1}^i(n_2)$. In practice the latter can be computed by letting

$$p_{n_1}^i(n_2) = a(n_2)\;\; p_{n_1}^i(0) + b(n_2)\;\; m^i(n_1) + c(n_2)\;\; l^i(n_1)$$

with $a(0) = 1$ and $b(0) = c(0) = 0$. The remaining $a(n_2)$, $b(n_2)$ and $c(n_2)$ are determined recurrently from (8), and the three unknowns $p^i(0)$, $m^i(n_1)$, $l^i(n_1)$, are computed using the normalization condition (9), eqns. (3) and (4). It is clear that if our scheme converges, i.e.

$$\lim_{i\to\infty}\; p_{n_1}^i(n_2) \text{ exists for all } n_1, n_2, \text{ we have}$$
$$\lim_{i\to\infty}\; p_{n_1}^i(n_2) = p_{n_1}(n_2).$$

4. Sufficient Convergence Conditions

Now we discuss sufficient convergence conditions. The sufficient convergence conditions under which our method converges can be obtained by proceeding in the same way as has been done by Brandwajn[1] for obtaining the sufficient convergence conditions for his method of solving balance equations of finite two-dimensional birth and death processes. For the above mentioned reason and also for reason of economizing space, the details of derivation of the sufficient convergence conditions are not being presented here.

5. Extension

There is scope to extend the present work. The method may be extended to the case where there are r repairmen instead of two repairmen each specializing in one type of machine stoppage with r kinds of stoppages in all.

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