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ON THE CHARACTERIZATION OF NONOSCILLATORY MOTIONS IN ROTATORY HYDROMAGNETIC THERMOHALINE CONVECTION

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Conditions for characterizing nonoscillatory motions which may be neutral or unstable for Veronis' [*J. Mar. Res.* 23 (1965), 1-17] and Stern's [*Tellus*, 12 (1960), 172-75] configurations with a uniform vertical rotation and magnetic field, are established. The results are uniformly valid for all combinations of free insulating (FI), rigid perfectly conducting (RPC) and rigid insulating (RI) boundaries.

1. INTRODUCTION

The establishment of the nonoccurrence of any slow oscillatory motions which may be neutral or unstable implies the validity of the principle of exchange of stabilities (PES). The validity of PES in a certain class of stability problems eliminates the unsteady terms from the linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and Southwell[1] proved the validity of PES for the classical Rayleigh-Bénard convection problem (RBCP). However, no such result existed for the hydromagnetic RBCP. Banerjee *et al.*[2,3] established such a criterion which has further been extended by Gupta *et al.*[4]. The aim of the present paper is to establish criteria for characterizing nonoscillatory motions which may be neutral or unstable for Veronis' and Stern's configurations[5,6] with a uniform vertical rotation and magnetic field. It is shown that for rotatory hydromagnetic Veronis' thermohaline convection, if

$$\frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \leqslant 1$$

then an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular the PES is valid, where R_s, T, Q, σ, σ_1 and τ respectively denote the salinity Rayleigh number, the Taylor number, the Chandrasekhar number, the thermal Prandtl number, the magnetic Prandtl number and the ratio of mass diffusivity to heat diffusivity. Similar results have been derived for rotatory hydromagnetic Stern's thermohaline convection. All these results are uniformly valid for all combinations of FI, RPC and RI boundaries and the results of Banerjee *et al.*[2,3], Gupta *et al.*[4] for hydromagnetic Rayleigh-Bénard convection and rotatory hydromagnetic Rayleigh-Bénard convection follow as a consequence.

2. Mathematical Analysis

The governing nondimensional linearized perturbation equations for Veronis' thermohaline convection problem in the presence of a uniform vertical rotation and magnetic field are (cf. Gupta *et al.*[7]):

$$(D^2 - a^2)\ (D^2 - a^2 - \frac{p}{\sigma})\ W = R_T\, a^2\, \theta - R_s\, a^2\, \Phi - Q\, D\, (D^2 - a^2)\, h_z + TDZ \qquad \ldots(1)$$

$$(D^2 - a^2 - p)\, \theta = -W \qquad \ldots(2)$$

$$(D^2 - a^2 - \frac{p}{\tau})\, \Phi = -\frac{W}{\tau} \qquad \ldots(3)$$

$$(D^2 - a^2 - \frac{p\sigma_1}{\sigma})\, h_z = -DW \qquad \ldots(4)$$

$$(D^2 - a^2 - \frac{p}{\sigma})\, Z = -DW - QDX \qquad \ldots(5)$$

$$(D^2 - a^2 - \frac{p\sigma_1}{\sigma})\, X = -DZ \qquad \ldots(6)$$

together with the boundary conditions

$$W = 0 = \theta = \Phi = D^2W = DZ = X = Dh_z \mp ah_z \text{ at } z = 0$$

and $z = 1$ (*FIB*)

or

$$W = 0 = \theta = \Phi = DW = Z = DX = h_z \text{ at } z = 0$$

and $z = 1$ (*RCB*)

or

$$W = 0 = \theta = \Phi = DW = Z = X = Dh_z \mp ah_z \text{ at } z = 0$$

$$\text{and } z = 1 \ (RIB) \qquad \ldots(7)$$

or upper boundary *FI* and lower boundary *RPC* without loss of generality.

The various symbols occurring in the above equations have the same meanings as in Gupta *et al.*[7]. Further the governing equations and boundary conditions for Stern's configuration with a uniform vertical rotation and magnetic field are given by eqns. (1) – (7) with $R_T < 0$ and $R_S < 0$.

We prove the following theorems :

Theorem 1—If $(p, W, \theta, \Phi, h_z, Z, X)$, $p = p_r + i\,p_i$, $p_r \geqslant 0$,

is a solution of eqns. (1) – (7) with $R_T > 0$, $R_S > 0$

and

$$\frac{R_S\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \leqslant 1$$

then $p_i = 0$. In particular,

$$p_r = 0 \Rightarrow p_i = 0 \text{ if } \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4} + \frac{Q\sigma_1}{\pi^2} \leqslant 1.$$

PROOF : Multiplying eqns. (1) – (6) by W^*, $-R_Ta^2\,\theta^*$, $\tau R_s a^2\Phi^*$, $-\frac{Q\,p^*\sigma_1}{\sigma}\,h_z^*$, TZ^* and $-TQX^*$ respectively (* indicates complex conjugation), integating over the range of z by parts appropriately utilizing (7) where necessary and then adding the resulting equations we have

$$J_1 + p\,J_2 + p^*\,J_3 = 0 \qquad \ldots(8)$$

where

$$J_1 = \int_0^1 |\,(D^2 - a^2)\,W\,|^2\,dz + Q\int_0^1 |\,DW\,|^2\,dz$$

$$+R_T\,a^2\int_0^1 (\,|\,D\theta|^2 + a^2\,|\,\theta\,|^2)\,dz - \tau R_s a^2\int_0^1 (\,|\,D\Phi\,|^2 + a^2\,|\,\Phi\,|^2)\,dz$$

(equation continued on p. 103)

$$+ \frac{Q \mid p \mid^2 \sigma_1^2}{\sigma^2} \int_0^1 \mid h_z \mid^2 dz - T \int_0^1 (\mid DZ \mid^2 + a^2 \mid Z \mid^2) dz$$

$$+ TQ \int_0^1 (\mid DX \mid^2 + a^2 \mid X \mid^2) dz + 2 \operatorname{Re} \Big[\int_0^1 (R_S a^2 \Phi W^*$$

$$- R_T a^2 \theta W^* + \frac{Q\sigma_1}{\sigma} pW^* Dh_z - TW^* DZ + TQZ^* DX) dz \Big],$$

$$J_2 = \frac{1}{\sigma} \int_0^1 (\mid DW \mid^2 + a^2 \mid W \mid^2) dz + R_T a^2 \int_0^1 \mid \theta \mid^2 dz + \frac{TQ\sigma_1}{\sigma} \int_0^1 \mid X \mid^2 dz$$

$$- R_S a^2 \int_0^1 \mid \Phi \mid^2 dz - \frac{T}{\sigma} \int_0^1 \mid Z \mid^2 dz$$

$$J_3 = \frac{Q\sigma_1}{\sigma} \Big[\Gamma + \int_0^1 (\mid D h_z \mid^2 + a^2 \mid h_z \mid^2) dz,$$

$$\Gamma = a [(\mid h_z \mid^2)_0 + (\mid h_z \mid^2)] \geqslant 0,$$

and 'Re' denotes the real part.

Imaginary part of eqn. (8) for $p_I \neq 0$ yields

$$\frac{1}{\sigma} \int_0^1 (\mid DW \mid^2 + a^2 \mid W \mid^2) dz + R_T a^2 \int_0^1 \mid \theta \mid^2 dz + \frac{TQ\sigma_1}{\sigma} \int_0^1 \mid X \mid^2 dz$$

$$= R_S a^2 \int_0^1 \mid \Phi \mid^2 dz + \frac{Q\sigma_1}{\sigma} \Big[\Gamma + \int_0^1 (\mid D h_z \mid^2 + a^2 \mid h_z \mid^2) dz \Big]$$

$$+ \frac{T}{\sigma} \int_0^1 \mid Z \mid^2 dz. \qquad \ldots(9)$$

Multiplying eqn. (3) by its complex conjungate, integrating over the range of z, and using boundary conditions (7), we have

$$\int_0^1 | D^2 \Phi |^2 dz + | (a^2 + \frac{p}{\tau}) |^2 \int_0^1 | \Phi |^2 dz + 2 (a^2 + \frac{p_r}{\tau}) \int_0^1 | D\Phi |^2 dz$$

$$= \frac{1}{\tau^2} \int_0^1 | W |^2 dz. \qquad ...(10)$$

Since $p_r \geqslant 0$, it follows from eqn. (10) that

$$2a^2 \int_0^1 | D \Phi |^2 d_z < \frac{1}{\tau^2} \int_0^1 | W |^2 dz$$

which upon using Poincare inequality[8] gives

$$2a^2 \pi^2 \int_0^1 | \Phi |^2 dz < \frac{1}{\tau^2 \pi^2} \int_0^1 | D W |^2 dz. \qquad ...(11)$$

Further as in Gupta *et al.*[4], it follows from eqns. (4)-(5) that

$$\Gamma + \int_0^1 (| D h_z |^2 + a^2 | h_z |^2) dz < \frac{1}{\pi^2} \int_0^1 | D W |^2 dz \qquad ...(12)$$

and

$$\int_0^1 | Z |^2 dz < \frac{1}{\pi^4} \int_0^1 | D W |^2 dz. \qquad ...(13)$$

Equation (9) together with inequalities (11)-(13) yields

$$\frac{1}{\sigma} \left[1 - \left(\frac{R_S \sigma}{2 \tau^2 \pi^2} + \frac{Q \sigma_1}{\pi^4} + \frac{T}{\pi^4} \right) \right] \int_0^1 | D W |^2 dz$$

(equation continued on p. 105)

$$+ \frac{a^2}{\sigma} \int_0^1 |W|^2 \, dz + R_T a^2 \int_0^1 |\theta|^2 \, dz + \frac{Q T \sigma_1}{\sigma}$$

$$\int_0^1 |X|^2 \, dz < 0. \qquad \ldots (14)$$

Inequality (14) clearly implies that

$$\frac{R_S \sigma}{2\tau^2 \pi^4} + \frac{Q \sigma_1}{\pi^2} + \frac{T}{\pi^4} > 1.$$

Hence

$$p_r \geqslant 0, \; p_i \neq 0 \Rightarrow \frac{R_S \sigma}{2\tau^2 \pi^4} + \frac{Q \sigma_1}{\pi^2} + \frac{T}{\pi^4} > 1 .$$

This proves the theorem.

The essential content of Theorem 1 from the point of view of linear stability theory is that for the problem of Veronis' thermohaline configuration with a uniform vertical rotation and magnetic field an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular PES is valid if

$$\frac{R_S \sigma}{2\tau^2 \pi^4} + \frac{Q\sigma_1}{\pi^2} + \frac{T}{\pi^4} \leqslant 1 .$$

Special Cases—It follows from Theorem 1 that an arbitrary neutral or unstable mode is nonoscillatory in character and in particular PES is valid for :

(i) Rayleigh-Bénard convection (RBC) ($T = 0 = Q = R_S$)

(Pellew and Southwell[1]);

(ii) rotatory RBC ($T = 0 = R_S$) if $\frac{T}{\pi^4} \leqslant 1$;

(iii) hydromagnetic RBC ($T = 0 = R_S$) if $\frac{Q\sigma_1}{\pi^4} \leqslant 1$ (Banerjee *et al.*[2])

(iv) rotatory hydromagnetic RBC ($R_S = 0$) if $\frac{Q \sigma_1}{\pi^2} + \frac{T}{\pi^4} \leqslant 1$

(Gupta *et al.*[4]);

(v) Veronis' thermohaline configuration (VTC) ($T = 0 = Q$)

if

$$\frac{R_S\,\sigma}{2\tau^2\,\pi^4} \leqslant 1;$$

(vi) rotatory VTC ($Q = 0$) if $\frac{R_S\,\sigma}{2\tau^2\,\pi^4} + \frac{T}{\pi^4} \leqslant 1$;

and

(vii) hydromagnetic VTC ($T = 0$) if $\frac{R_S\,\sigma}{2\tau^2\,\pi^2} + \frac{Q\,\sigma_1}{\pi^2} \leqslant 1$.

Theorem 2—If $(p, W, \theta, \Phi, h_z, Z, X)$, $p = p_r + ip_i$, $p_r \geqslant 0$, is a solution of eqns. (1) – (7) with $R_T < 0$, $R_S < 0$ and $\frac{|R_T|\,\sigma}{2\pi^4} + \frac{T}{\pi^4} + \frac{Q\,\sigma_1}{\pi^2} \leqslant 1$ then $p_i = 0$. In particular, $p_r = 0 \Rightarrow p_i = 0$ if $\frac{|R_T|\,\sigma}{2\pi^4} + \frac{T}{\pi^4} + \frac{Q\,\sigma_1}{\pi^2} \leqslant 1$.

PROOF : Putting $R_T = -|R_T|$, $R_S = -|R_S|$ in eqns. (1) – (6), proceeding as in Theorem 1, using the inequalities (12), (13) and

$$2a^2 \int_0^1 |\theta|^2\,dz < \frac{1}{\pi^4}\int_0^1 |DW|^2\,dz$$

which is derived from eqn. (2) in a manner similar to the derivation of inequality (11), we get the result. The essential content of Theorem 2 from the point of view of linear stability is similar to that of Theorem 1.

Special Cases—It follows from Theorem 2 that an arbitrary neutral or unstable mode is nonscillatory in character and in particular PES is valid for

(i) Stern's thermohaline configuration (STC) ($T = 0 = Q$) if

$$\frac{|R_T|\,\sigma}{2\pi^4} \leqslant 1;$$

(ii) Rotatory STC ($Q = 0$) if $\frac{|R_T|\,\sigma}{2\pi^4} + \frac{T}{\pi^4} \leqslant 1$;

and

(iii) Hydromagnetic STC ($T = 0$) if $\frac{|R_T|\,\sigma}{2\pi^4} + \frac{Q\,\sigma_1}{\pi^2} \leqslant 1$.

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