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A MATRIX GENERALIZATION OF A VECTOR MAXIMIZATION PROBLEM

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Conditions are derived to relate proper efficiency of a given solution for a matrix-valued function to the stability of certain multiple objective maximization problem. Our results generalize the corresponding results of Benson and Morin[1] on vector maximization problem relating properly efficient solutions to stability of single objective maximization problem. We also give examples of vector maximization problems for matrices with properly efficient solutions.

1. INTRODUCTION

Vector maximization problems are encountered when $p (\geq 2)$ non-comparable criterion functions are to be simultaneously maximized over a given feasibility region. The concept of efficiency has played a useful role in analysis of such problems. Benson and Morin[1] discussed in detail such problems and gave necessary and sufficient conditions for an efficient solution to a vector maximization problem to be properly efficient. These conditions relate the proper efficiency of a given solution to the stability of certain single objective maximization problems. In the present work we extend the above result of Benson and Morin[1] for properly efficient solution to a maximization problem with a matrix-valued objective function. We also give an example to support our extension.

2. THE VECTOR MAXIMIZATION PROBLEM AND EXTENSIONS WITH BASIC DEFINITIONS AND PRELIMINARIES

Consider a vector valued criterion function

$$f(x) = [f_1(x), f_2(x), \ldots f_p(x)]$$

defined over a set X given by

$$X = \{x \in R^n \mid g_i(x) \geq 0, \text{ all } i \in I\}$$

where $p \geqslant 2$, $f_j : R^n \to R \ \forall\ j \in J = \{1, 2, \ldots, p\}$, $I = \{1, 2, \ldots, m\}$,

and $g_i : R^n \to R \ \forall i \in I$. Let $R^n_+ = \{x \in R^n \mid x_l \geqslant 0, \ l = 1, 2, \ldots n\}$

and for any $k \in J$, let J_k denote the set $(J - \{k\})$.

Then, the vector maximization problem

(P) VMAX : $f(x)$ subject to $x \in X$

is the problem of finding all solutions that are efficient in the sense of Definition 2.1.

Definition 2.1—$\bar{x}$ is said to be an efficient solution of (P) if $\bar{x} \in X$ and $f_i(x) > f_i(\bar{x})$ for some $x \in X$ and some $i \in J$ implies that there exists at least one $j \in J_i$ such that $f_i(x) < f_j(\bar{x})$.

Definition 2.2—$\bar{x}$ is said to be a 'properly efficient' solution of (P) when it is efficient for (P) and there exists a scalar $M > 0$ such that for each $i \in J$ and each $x \in X$ satisfying

$$f_i(x) > f_j(\bar{x}), \quad \text{there exists at least one } j \in J \text{ with}$$

$$f_j(x) < f_j(\bar{x}) \text{ and } [f_i(x) - f_i(\bar{x})]/[f_j(\bar{x}) - f_j(x)] \leqslant M.$$

The above definition of properly efficient solution is due to Geoffrion[2]. For each properly efficient solution of (P), given any criterion, the marginal gain in that criterion relative to a loss in some other criterion is bounded from above. An efficient solution that is not properly efficient is said to be improperly efficient. We intend to replace the problem (P') given as follows. Let

$$F(x) = \begin{bmatrix} f_1^1(x) & f_2^2(x) \ldots f_1^m(x) \\ f_2^1(x) & f_2^2(x) \ldots f_2^m(x) \\ f_p^1(x) & f_p^2(x) \ldots f_p^m(x) \end{bmatrix}$$

be defined over a set X given by

$$X = \{x \in R^n \mid g_s(x) \geqslant 0, \ \forall\ s \in I\}$$

where $f_i^j : X \subseteq R^n \to R \ \forall\ j \in J = \{1, 2, \ldots p\}$ and $\forall\ i \in Q = \{1, 2, \ldots m\}$,

$I = \{1, 2, \ldots q\}$, and for any $k \in J$, let $J_k = (J - \{k\})$.

Then the matrix maximization problem

(P′) MMAX : $F(x)$ subject to $x \in X$

is the problem of finding all solutions that are efficient in the modified sense given in Definition 2.3.

Definition 2.3—$\bar{x}$ is said to be 'efficient solution' of (P′) if $\bar{x} \in X$ and $f^i_{j_i}(x) > f^i_{j_i}(\bar{x})$, for each i and some $x \in X$ and some $j_i \in J$ implies that there exists at least one $l_i \in J_{j_i}$ such that $f^i_{j_i}(x) < f^i_{l_i}(\bar{x})$.

Definition 2.4—$\bar{x}$ is said to be a 'properly efficient' solution of (P′) when it is efficient for (P′) and there exists scalars $M_i > 0$ such that for each $j_i \in J$ and each $x \in X$ satisfying $f^i_{j_i}(x) > f^i_{j_i}(\bar{x})$ for each i, there exist at least one $l_i \in J_{j_i}$ with $f^i_{l_i}(x) < f^i_{l_i}(\bar{x})$ and $\left[f^i_{j_i}(x) - f^i_{j_i}(\bar{x})\right] \Big/ \left[f^i_{l_i}(\bar{x}) - f^i_{l_i}(x)\right] \leqslant M_i$.

The following definitions will aid in our derivation of necessary and sufficient conditions for an efficient solution to be properly efficient.

Definition 2.5—$\bar{x}$ is said to be a $(k_1, k_2 \ldots, k_m)$ th-entry-efficient solution of (P′), where $k_i \in J$ $(i = 1, \ldots m)$, if $\bar{x} \in X$ and for each i, $f^i_{k_i}(x) > f^i_{k_i}(\bar{x})$ for some $x \in X$ implies that there exists at least one $j_i \in J_{k_i}$ such that $f^i_{j_i}(x) < f^i_{j_i}(\bar{x})$ for each i.

Definition 2.6—$\bar{x}$ is said to be a properly $(k_1, k_2, \ldots k_m)$ th-entry efficient solution of (P′), where $k_i \in J$ when it is $(k_1, k_2, \ldots, k_m)$ th-entry efficient for (P′) and there exists a scalar $M_{k_i} > 0$ (for each i) such that for each $x \in X$ satisfing $f^i_{k_i}(x) > f^i_{k_i}(\bar{x})$, there exists at least one $j_i \in J_{k_i}$ with $f^i_{j_i}(x) < f^i_{j_i}(\bar{x})$ and $\left[f^i_{k_i}(x) - f^i_{k_i}(\bar{x})\right] \Big/ \left[f^i_{j_i}(\bar{x}) - f^i_{j_i}(x)\right] \leqslant M_{k_i}$.

The results given below would follow immediately from the definitions.

Proposition 2.1—A point $\bar{x}$ is an efficient solution of (P′) if and only if it is a $(k_1, k_2, \ldots, k_m)$-th-entry efficient solution of (P′) for each $(k_1, k_2, \ldots k_m)$ tuple in $J \times J \times \ldots \times J$.

Proposition 2.2—A point $\bar{x}$ is a properly efficient solution of (P') if and only if it is a properly $(k_1, k_2, \ldots k_m)$ th-entry-efficient solution of (P') for each tuple $(k_1, k_2, \ldots k_m) \in J \times J \times \ldots \times J$.

3. Necessary and Sufficient Conditions for Proper Efficiency

Consider the multiple objective problem for each i,

$$P_u^i: \max f_{k_i}^i(x) + \sum_{j_i \in J_{k_i}} u_{j_i}^i \; f_{j_i}^i(x)$$

subject to $x \in X$.

Let $\bar{X}_u^i$ denote the optimal solutions for (P_u^i). The following lemma characterizing the $(k_1, k_2, \ldots, k_m)$ th-entry properly efficient solution for (P') in terms of solutions of (P_u^i), is generalization of Lemma 1 of Benson and Morin[1].

Lemma 3.1— (i) Let $\bar{x} \in \bigcap_{i=1}^{m} \bar{X}_{u^i}$ for some $\bar{u}^i \in R_+^{p-1}$, $i = 1, 2, 3, \ldots, m$,

then $\bar{x}$ is a properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient solution for (P').

(ii) Suppose that the entries in the matrix of $F(x)$ are all concave functions on the convex set X [Note that the set X is convex only when $g_i(\,.\,)$ are concave functions for each i]. Then $\bar{x}$ is a properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient solution for (P') if and only if there exists a tuple $\bar{u}^i \in R_+^{p-1}$ $(i = 1, 2, \ldots m)$ such that $\bar{X} \in \bigcap_{j=1}^{m} \bar{X}_{\bar{u}^i}$.

Proof : *Sufficiency* : Suppose there exists a tuple $\bar{u}^i \in R_+^{p-1}$ $(i = 1, 2, \ldots m)$ such that $\bar{x} \in \bigcap_{j=1}^{m} \bar{X}_{\bar{u}^i}$. Let $U_{k_i}^i = \{j_i \mid \bar{u}_{j_i}^i > 0, j_i \in J_{k_i}\}$. Then either (a) $U_{k_i} = \Phi$ or

(b) $U_{k_i}^i \neq \Phi$.

Case 1—$U_{k_i}^i = \Phi$. Then $\bar{u}_{j_i}^i = 0$ for all $j_i \in J_{k_i}$. So $\bar{x}$ achieves the global maximum for $f_{k_i}^i(x)$ over X. Thus there is no $x \in X$ such that $f_{k_i}^i(x) > f_{k_i}^i(\bar{x})$.

Since this is true for each i, by Definition 2.5 it follows that $\bar{x}$ is $(k_1, k_2, \ldots, k_m)$ th-entry efficient solution for (P').

Case 2— $U^l_{k_l} = \Phi$. Suppose $x \in X$ and $f^l_{k_l}(x) > f^l_{k_l}(\bar{x})$. Since $\bar{x}$ is an optimal solution to (P_{u^l}) with $u^l = \bar{u}^l$, we have

$$f^l_{k_l}(x) + \sum_{j_l \in J_{k_l}} \bar{u}^1_{j_l} f^l_{j_l}(x) \leqslant f^l_{k_l}(\bar{x}) + \sum_{j_l \in J_{k_l}} \bar{u}^l_{j_l} f^l_{j_l}(\bar{x}).$$

By definition of $U^l_{k_l}$, this can be written as

$$f^l_{k_l}(x) + \sum_{j_l \in U^{l_k}_{i}} \bar{u}^l_{j_l} f^l_{j_l}(x) \leqslant f^l_{k_l}(\bar{x}) + \sum_{j_l \in U^{l_k}_{i}} u^l_{j_l} f^l_{j_l}(\bar{x}).$$

Since $f^l_{k_l}(x) > f^l_{k_l}(\bar{x})$, we have

$$\sum_{j_l \in U^{l_k}_{i}} \bar{u}^l_{j_l} f^l_{j_l}(x) < \sum_{j_l \in U^{l_k}_{i}} \bar{u}^l_{j_l} f^l_{j_l}(\bar{x}).$$

That is

$$\sum_{j_l \in U^l_k} \bar{u}^l_{j_l} [f^l_{j_l}(x) - f^l_{j_l}(\bar{x})] < 0.$$

Since $\bar{u}^l_{j_l} > 0$ for all $j_l \in U^l_k$, it is necessary that $f^l_{j_l}(x) - f^l_{j_l}(\bar{x}) < 0$ for at least one $j_l \in U^l_{k_l}$. Thus $f^l_{j_l}(x) < f^l_{j_l}(\bar{x})$ for some $j_l \in J_{k_l}$. By Definition 2.5, $\bar{x}$ is $(k_1, k_2, \ldots, k_m)$ th-entry efficient.

Now we show that $\bar{x}$ is actually properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient for (P'), with $M_{k_l} = (P - 1) \max \{\bar{u}^l_{j_l} j_l \in J_{k_l}\}$, for each i. Assume to the contraryt hat there exists an $x \in X$ such that $f^l_{k_l}(x) > f^l_{k_l}(\bar{x})$ for each i and

$f^l_{k_l}(x) - f^l_{k_l}(\bar{x}) > M_{k_l}(f^l_{j_l}(\bar{x}) - f^l_{j_l}(x))$ for all $j_l \in J_{k_l}$, such

that $f^l_{j_l}(x) < f^l_{j_l}(\bar{x})$. By definition of M_{k_l}, we have

$$f^{l}_{j_l}(x) - f^{l}_{k_l}(\bar{x}) > (p-1)\ \bar{u}^{1}_{j_l}\left[f^{l}_{j_1}(\bar{x}) - f^{l}_{j_l}(x)\right] \text{ for all } j_l \in J_{k_l}.$$

The last inequality holds even for those $j_l \in J_{k_l}$ such that $f^{l}_{j_l}(x) \geqslant f^{l}_{j_l}(\bar{x})$ because, in these cases, the right-hand side is nonpositive, and in all cases, the left hand side is positive. Dividing both sides by $(p-1)$ and summing over all J_{k_i} we obtain,

$$f^{l}_{k_i}(x) - f^{l}_{k_l}(\bar{x}) > \sum_{j_l \in J_{k_l}} \bar{u}^{l}_{j_l}\left[\ f^{i}_{j_i}(\bar{x}) - f^{l}_{j_i}(x)\ \right]$$

which upon rearranging becomes

$$f^{l}_{k_l}(x) + \sum_{j_l \in J_{k_l}} \bar{u}^{l}_{j_l}\ f^{i}_{j_i}(x) > f^{l}_{k_l}(\bar{x}) + \sum_{j_l \in J_{k_l}} \bar{u}^{l}_{j_l}\ f^{l}_{k_l}(\bar{x}).$$

This contradicts $\bar{x} \in \bigcap_{i=1}^{m} \bar{X}_{\bar{u}^i}$ and completes the proof of (i).

(ii) The proof of the if part in this statment is provided in the proof of (i) above.

Assume $\bar{x}$ is a properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient solutions for (P'). Then the following system of inequalities for each i admits no solution in X :

$$f^{l}_{k_l}(x) - f^{l}_{k_l}(\bar{x}) > 0$$

$$f^{l}_{k_l}(x) + M_{k_l}\ f^{i}_{j_i}\ x - f^{i}_{k_l}(\bar{x}) - M_{k_l}\ f^{l}_{j_l}(\bar{x}) > 0 \ \forall\ i_j \in J_{k_i}.$$

From here onwards we proceed as in Benson and Morin[1] using generalized Gordon's theorem[4], to show that $\bar{x} \in \bar{X}_{\bar{u}^i}$, for each i, where $\bar{u}^{l}_{j_l} = M_{k_l}\ u^{l}_{j_l}\ /C$, where $C = \sum_{j_l=1}^{p} u^{l}_{j_l}$. Since $M_{k_l} > 0$, $C > 0$ and $u^{i}_{j_i} > 0$ for each i, hence $\bar{u}^i \in R^{p-1}_{+}$ for $i = 1, 2, 3, \ldots m$, and the proof of (ii) is complete.

With the aid of previous lemma one can give the necessary and sufficient conditions for a $(k_1, k_2, \ldots, k_m)$ th-entry- efficient solution for (P') to be properly $(k_1, k_2 \ldots, k_m)$ th-entry efficient. Consider the problem, for each i,

$$\left(P_{b^i, k_i} \right) : \max f^i_{k_i}(x)$$

subject to $f^i_{j_i}(x) - b^i_{j_i} \geqslant 0, \ \forall\, j_i \in J_{k_i}$

$$x \in X,$$

where $k_i \in J$ and $\left\{ b^i_{j_i} \,\middle|\, j_i \in J_{k_i} \right\}$ is any set of $p-1$ numbers for each fixed i. The following definition is adapted from Geoffrion[2] for perturbation function.

Definition 3.2—The perturbation function $v^i : R^{p-1} \to R$ associated with (P_{b^i, k_i}), for each i, is defined as

$$v^i(y^i) = \sup_{x \in X} \left\{ f^i_{k_i}(x) \,\middle|\, f^i_{j_i}(x) - b^i_{j_i} \geqslant y^i_{j_i}, \ \forall\, j_i \in J_{k_i} \right\}.$$

Note that whenever the optimal value of (P_{b^i, k_i}) exists for each i, it is equal to $v^i(0)$.

Definition 3.3— (P_{b^i, k_i}) is said to be stable for each i $(i = 1, 2, \ldots m)$, when $v^i(0)$ is finite and there exists a scalar $M_i > 0$, such that, for all $y^i \neq 0$, $[v^i(y^i) - v^i(0)]/\|y_i\| \leqslant M$. If $(P_{b^i k_i})$ is not stable for each i, the ratio of the improvement in it's optimal value to the amount of perturbation can be made as large desired. The choice of the particular norm $\| \cdot \|$ used to measure the amount of perturbation is arbitrary.

Lemma 3.4—Assume that the matrix corresponding to $F(x)$ has entries consisting of concave functions on the nonempty convex set X. Suppose $\bar{x}$ is a $(k_1, k_2, \ldots, k_m)$ th-entry efficient solution for (P'). Then $\bar{x}$ is a properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient soluiion for (P') if and only if $(P_{\bar{b}^i_{k_i}})$, where $\bar{b}^i_{j_i} = f^i_{j_i}(\bar{x})$, $\forall\, j_i \in J_{k_i}$, is stable for each i $(i = 1, 2, \ldots, m)$.

PROOF : One has to define for each i $(i = 1, 2, \ldots, m)$ the dual of the problem $(P_{\bar{b}^i, k_i})$ as $(D_{\bar{b}^i, k_i})$ given in the following manner

$$\left(D_{\bar{b}^i, k_i} \right) \Phi_{k_i} = \min_{u^i_{j_i} \geqslant 0} \left\{ \max_{x \in X} \left[f^i_{k_i}(x) + \sum_{j_i \in J_{k_i}} u^i_{j_i} \left(f^i_{j_i}(x) - f^i_{j_i}(\bar{x}) \right) \right] \right.$$

where $u^l_{j_l}$ is a $(p-1)$-vector of dual variables, then the proof follows the analogous course as in Benson and Morin[1].

Our main result is stated as follows :

Theorem 3.5—Assume that the entries of the matrix of $F(x)$ are concavefu nctions on the nonempty convex set X. Suppose $\bar{x}$ is an efficient solution of (P'). Then $\bar{x}$ is a properly efficient solution for (P') if and only if $(P_{\bar{b}^i, k_l})$, where $\bar{b}^l_{j_l} = f^l_{j_l}(\bar{x})$, $\forall\, ^l_j \in J_{k_l}$, is stable for each k_l $(k_1, k_2, \ldots, k_m)$ in J.

PROOF : By Proposition 2. 1, $\bar{x}$ is $(k_1, k_2, \ldots k_m)$ th-entry efficient solution for each tuple $(k_1, k_2, \ldots, k_m) \in J \times J \times \ldots \times J$. By Lemma 3.4, then $\bar{x}$ is properly k-th--entry efficient for each $(k_1, k_2, \ldots, k_m)$ if and only if $(P_{\bar{b}^i, k_l})$ is stable for each $k^l \in J$ $(i = 1, 2, 3, \ldots, m)$. Again by Proposition 2.2, $\bar{x}$ is properly $(k_1, k_2, \ldots, k_m)$ th-entry efficient for each $(k_1, k_2, \ldots k_m)$ if and only if $\bar{x}$ is properly efficient. Combining two statements we get the desired conclusion.

Following corollary is obtained easily as in Benson and Morin[1] for our matrix generalization from Theorem 3.5.

Corollary 3.6—Assume $f^l_j(x)$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, p)$ and $g_l(x)$ $i = 1, 2, \ldots g)$ are linear functions. Then any efficient solution $\bar{x}$ for (P') is properly efficient solution for (P').

Example —Consider the problem :

$$(P_1) \quad M\,MAX : F(x) = \begin{bmatrix} -x^2 + 4 & -(x-1)^4 \\ -(x-1)_4 & -x^2 + 4 \end{bmatrix}$$

Subject to $x + 100 \geqslant 0$.

The above example is modified from of the example given in Benson and Morin[1]. The set of efficient points for $(P\ 1)$ is $\{x \mid 0 \leqslant x \leqslant 1\}$.

Let $E^p = \{x \mid 0 < x < 1\}$. Suppose $\bar{x} \in E^p$. Theorem 3.5 can be invoked to show that $\bar{x}$ is properly efficient. Here $f^1_1(x) = -x^2 + 4$, $f^1_2(x) = -(x-1)^4$, $f^2_1(x) = -(x-1)^4$, and $f^2_2(x) = -x^2 + 4$ are all concave on the nonempty convex set $X = \{x \in R \mid x + 100 \geqslant 0\}$. For $k_1 = 1$, $x = \bar{x}/2 + \frac{1}{2}$ satisfies $g_1(x) \geqslant 0$,

$f_2^1(x) > f_2^1(\bar{x})$. For $k_1 = 2$, $x = \bar{x}/2$ satisfies $g_1(x) \geqslant 0$, $f_1^1(x) > f_1^1(\bar{x})$. Thus Slater's constraint qualification holds for $\bar{X}_1^1 = \{x \in X \mid f_2^1(x) - f_2^1(\bar{x}) \geqslant 0\}$ and for $\bar{X}_2^1 = \{x \in X \mid f_1^1(x) - f_1^1(\bar{x}) \geqslant 0\}$. This implies that the problems {max $f_{k_i}^i(x) \mid x \in \bar{x}_{k_i}^1, k_1 = 1, 2\}$ are both stable. Similary one can show that max $\{f_{k_i}^1(x) \mid x \in X_{k_2}^2, k_2 = 1, 2\}$ are both stable. Hence, by Theorem 3.5, $\bar{x}$ is properly efficient for $(P1)$.

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