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LIMIT MODIFICATIONS IN OPTIMIZATION AND THEIR APPLICATIONS TO SADDLE POINT PROBLEMS

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A multiplicative type of modification in optimization problem inf $\{f(x): x \in A\}$ with a sequence of functions $\{H_n(x)\}_{n\in N}$ is discussed in the present work to yield the problem inf $\{f(x)\, H_n(H): x \in A\}$. We discuss the convergnce properties for this class of modifications. The results obtained are applied to modify saddle point problems with partial modifications. It turns out that the present modification is yet another type of modification differing from the types already discussed by Eehar[1]. We apply our results for convex-concave saddle point problems as well with total modification.

1. INTRODUCTION

Let E be a locally convex linear topological vector space. We denote by $CL(E)$, the space of lower semi-continuous functions defined on E with values on $\bar{\mathbf{R}}$ (the extended real line) and by $\Gamma_0(E)$ the space of functions in $CL(E)$ which are convex, not identically equal to $+\infty$, and never taking on the value $-\infty$.

Our study is about the general optimization problem :

$$\{P\}\ \alpha = \inf\{f(x) : x \in A\},\ \Omega = \{x \in A : f(x) = \alpha\} \qquad ...(1.1)$$

where $f \in CL(E)$ and A is a nonempty closed subset of E. Consider a sequence of functions $\{f_n\}$ and the problems :

$$\{P_n\}\ \alpha_n = \inf\{f_n(x) : x \in A\},\ \Omega_n = \{x \in A : f_n(x) = \alpha_n\}. \qquad ...(1.2)$$

Definition 1.1—Any sequence $\{x_n\}$ such that for each $n \in N$, $x_n \in \Omega_n$ will be called a sequence of solutions of the family $\{P_n\}$.

Definition 1.2—The sequence of problems $\{P_n\}$ is called convergent to $\{P\}$ if $\Omega_n \neq \phi$, for each $n \in N$, $\lim_{n\to\infty} \alpha_n = \alpha$, and every sequence of solutions of the family $\{P_n\}$ has at least on cluster point and all the cluster points are solutions of $\{P\}$.

Definition 1.3—When the functions $\{f_n\}$ have the form $f_n = fH_n$, where $H_n \in CL(E)$, the sequence of problems $\{P_n\}$ will be called a $\{H_n\}$-multiplicative modification of $\{P\}$.

Remark : Behar[1] studied modifications in which $f_n = f + H_n$, $H_n \in CL(E)$. We follow the notation and terminologies used by Behar[1] in all our discussion throughout the paper. We treat below a special type of modification given below with properties of H_n stated explicitly.

2. One-Limit Modification

Consider a sequence of functions $H_n(x) = 1 + G_n(x)$ with $G_n(x) \in CL(E)$ satisfying the following assumptions :

(1) For every $n \in N$, $\exists\ M_n \in \bar{\mathbf{R}}$ such that

$$G_n(x) \geqslant M_n \text{ for each } x \in A$$

$$\lim_{n\to\infty} M_n = 0.$$

(2) For every $n \in N$, $\gamma_n > 0$ such that $\lim_{n\to\infty} \gamma_n = \infty$.

(3) The set of functions $\{h_n \in CL(E) : h_n = \gamma_n G_n\}$ is equicontinuous with respect to A. The function h, defined by $h(x) = \lim_{n\to\infty} h_n(x)$, is finite for all $x \in A$.

(4) The family of functions $\{h_n : n \in N\}$ is uniformly lower compact on A (i. e., for $\lambda \in \bar{\mathbf{R}}$, $\exists$ a compact $K_\lambda \subset A$ such that the sets $\{x \in A : h_n(x) \leqslant \lambda\}$ are included in K_λ for all $n \in N$.

Definition 2.1—If a sequence $\{H_n\}$ satisfies the assumption (3), then the sequence of problems $\{P_n\}$ defined by (1.2) is called a one Limit modification of the problem $\{P\}$.

Theorem 2.1—If α is finite and $\{P_n\}$ is a One-Limit modification of the problem $\{P\}$ then (1) $\Omega_n \neq \phi$ for all $n \in N$, (2) $\lim_{n\to\infty} \alpha_n = \alpha$; (3) if $\{x_n\}$ is a sequence of solutions then $\lim_{n\to\infty} f(x_n) = \alpha$ and $\{x_n\}$ has cluster points iff $\Omega \neq \phi$. In this case $\{P_n\}$ converges to $\{P\}$; (4) If $\bar{x}$ is a cluster point of a sequence of solutions then $h(\bar{x}) = \min\{h(z) : z \in \Omega\}$.

Proof : (1) Because of our assumptions, the function $f_n = fH_n = f(1 + G_n)$ is lower compact and bounded below on A, therefore $\Omega_n \neq \phi$.

(2) By definition of M_n,

$$f(x)(1 + M_n) \lneqq f(x)(1 + G_n(x)) \text{ for all } x \in A,\ n \in N,$$

hence

$$\alpha + \alpha M_n \lneqq f(x)(1 + G_n(x)) \quad \text{for all } x \in A,\ n \in N,$$

then

$$\alpha \leqslant \liminf \alpha_n \leqslant \limsup \alpha_n \lneqq f(x) \text{ for all } x \in A.$$

Therefore $\lim_{n\to\infty} \alpha_n = \alpha$.

(3) We have

$$\alpha + \alpha M_n \leqslant f(x_n)(1 + M_n) \lneqq (f x_n)(1 + G_n(x_n)) = \alpha_n.$$

Since $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} M_n = 0$, therefore we conclude that $\lim_{n\to\infty} f(x_n) = \alpha$. If $\bar{x}$ is a cluster point of $\{x_n\}$, the lower semi-continuity of f implies that $f(\bar{x}) \leqslant \lim_{n\to\infty} f(x_n) = \alpha$, which is possible only if $\bar{x} \in \Omega$. Conversely, if $\Omega \neq \phi$, let $u \in \Omega_n$ and $z \in \Omega$.

We have,

$$\alpha_n = f(u)(1 + G_n(u)) = f(u)\left(1 + \frac{1}{\gamma_n} h_n(u)\right)$$

$$\leqslant f(z)\left(1 + \frac{1}{\gamma_n} h_n(z)\right).$$

Since $f(z) \leqslant f(u)$ and $\gamma_n > 0$, we get

$$h_n(u) \lneqq h_n(z) \text{ for all } u \in \Omega_n,\ z \in \Omega,\ n \in N. \qquad \ldots(2.1)$$

Since the sequence $\{h_n(z)\}$ is convergent to $\{h(z)\}$ therefore there exists $\lambda \in N$ such that

$$h_n(u) \leqslant \lambda \text{ for all } u \in \Omega_n,\ n \in N.$$

The uniform lower compactness of the functions $\{h_n\}$ implies that there exists a compact set $K \subset A$ such that $\Omega_n \subset K$ for all $n \in N$, which implies in turn, that every sequence of functions has at least one cluster point. Since $\lim_{n\to\infty} f(x_n) = \alpha$, such a cluster point belongs to Ω.

(4) From equicontinuity of the family we get the result $h(\bar{x}) = \min\{h(z) : z \in \Omega\}$ from (2.1).

Remark 2.1 : For the rate of convergence we have the formula in this case given as follows :

If $M_n \geqslant 0$ and $z \in \Omega$, then

$$\alpha + \alpha M_n \leqslant \alpha_n \leqslant f(z)\left(1 + \frac{1}{\gamma_n} h_n(z)\right)$$

hence

$$0 \leqslant M_n \leqslant \frac{\alpha_n - \alpha}{\alpha} \leqslant \frac{1}{\gamma_n} h_n(z).$$

For the convex case the following analogous[1] remark is valid.

Remark 2.2 : Suppose the problem $\{P\}$ is convex (i. e., $f \in \Gamma_0(E)$ and A is a nonempty closed convex subset of E). In this case Ω is convex. We can choose the sequence $\{G_n\}$ such that the functions $\{h_n\}$ and h are strictly convex. In such a situation every problem in the sequence $\{P_n\}$ has a unique solution, so there is a unique sequence of solutions $\{x_n\}$. On the other hand any cluster point of $\{x_n\}$ is also the unique solution of the problem $\inf\{h(z): z \in \Omega\}$. Therefore the sequence $\{x_n\}$ is convergent.

3. Convex-Concave Saddle Point Problems

Let E and F be locally convex topological vector spaces and A and B nonempty closed convex subsets of E and F respectively. We consider a function $L : E \times F \to \bar{\mathbf{R}}$ satisfying.

(1) $L(., v) \in \Gamma_0(E)$ for all $v \in B$; $-L(x,.) \in \Gamma_0(F)$ for all $x \in A$.

(2) $\exists$ an $a_0 \in A$ and $b_0 \in B$ such that the following conditions are satisfied :

(i) $\inf\{L(x, b_0) : x \in A\} > -\infty$ and

(ii) $\sup\{L(a_0, v) : v \in B\} < \infty$.

(3) $\exists\, b \in B$ such that $L(., b)$ is lower compact on A.

(4) $\exists\, a \in A$ such that $L(a,.)$ is upper compact on B.

Consider the problems :

$$\{P\} \quad \alpha = \inf \{f(x) : x \in E\}, \quad \Omega_P = \{x \in A : f(x) = \alpha\} \qquad ...(3.1)$$

$$\{D\} \quad \beta = \sup \{g(v) : v \in F\}, \quad \Omega_D = \{v \in B : g(v) = \beta\} \qquad ...(3.2)$$

where,

$$f(x) = \begin{cases} \sup L(x, v) & \text{if } x \in A, \\ v \in B & \\ + \infty \text{ if } x \notin A; & \end{cases} \qquad ...(3.3)$$

$$g(v) = \begin{cases} \inf L(x, v) & \text{if } v \in B, \\ - \infty \text{ if } v \notin B. & \end{cases} \qquad ...(3.4)$$

When $\alpha = \beta$, α is called the saddle value of the triplet (L, A, B). We consider the problem : $\{S\}$ find the saddle value and saddle point $(\bar{x}, \bar{v})$ of the triplet (L, A, B) (i. e., $L(\bar{x}, v) \leqslant L(\bar{x}, \bar{v}) \leqslant L(x, \bar{v})$ for all $x \in A$, $v \in B$).

Problems $\{P\}$ and $\{D\}$ are called respectively, the primal and dual problems associated with the problem $\{S\}$. It is well-known that if the triplet (L, A, B) has a saddle point $(\bar{x}, \bar{v})$ then $\alpha = \beta = L(\bar{x}, \bar{v})$ and $\bar{x} \in \Omega_P$, $\bar{v} \in \Omega_D$. Conversely, if $\Omega_P \times \Omega_D \neq \phi$ and $\alpha = B$, then any element of $\Omega_P \times \Omega_D$ is a saddle point of (L, A, B). The following theorem is proved in Behar[1] with regard to existence of saddle point of (L, A, B).

Theorem 3.1—If the conditions 1, 2 and 3 are satisfied then the triplet (L, A, B) has a saddle value (i. e., $\alpha = \beta$), $\Omega_P \neq \phi$. If the condition 4 is also satisfied then (L, A, B) has at least one saddle point. We aim at the partial one-limit modification in the following way.

4. Partial One-Limit Modification

Consider a sequence of functions $H_n \in \Gamma_0(F)$ satisfying :

(a) Assumption 3 of section 2 with respect to nonempty closed convex subset B of F.

(b) The functions $h_n = \left(\frac{1}{\gamma_n}\right) G_n$ and $h = \lim_{n \to \infty} h_n$ are strictly convex.

(c) Each G_n is bounded away from 1 for $n \in N$.

The problem $\{S_n\}$ is given as follows :

$\{S_n\}$ find the saddle value and the saddle points of the triplet (L_n, A, B), where

$$L_n(x, v) = L(x, v) H_n(v) = L(x, v) (1 - G_n(v)).$$

Remark 4.1 : By this modification of the problem $\{S\}$, the problems $\{P\}$ and $\{D\}$ are also modified. The objective functions of the primal problem $\{P_n\}$ and dual problem $\{D_n\}$ associated with $\{S_n\}$ are :

$$f_n(x) = \begin{cases} \sup \{L_n(x, v) : v \in B\} \text{ if } x \in A, \\ +\infty \text{ if } x \notin A; \end{cases} \quad \ldots(4.1)$$

$$g_n(v) = \begin{cases} \inf \{L_n(x, v) : x \in A\} \text{ if } v \in B, \\ -\infty \text{ if } v \notin B, \end{cases}$$

$$= g(v)\, H_n(v) = g(v)\,(1 - G_n(v)) \quad \ldots(4.2)$$

where $g(v)$ is defined as in (3.4). Therefore the problem $\{D_n\}$ is an one-limit modification of $\{D\}$ and the results of Section 2 can be applied to the family of problems $\{D_n\}$. We have the following result.

Theorem 4.1—If the triplet (L, A, B) satisfies conditions 1-4 of Section 3 and the sequence $\{H_n(x)\}$ satisfies assumptions a, b and c of Section 4 then the triplet (L, A, B) has a saddle value α and

(1) For every $n \in N$, the triplet (L_n, A, B) has at least one saddle point.

(2) If α_n is the saddle value of (L_n, A, B), then $\lim_{n \to \infty} \alpha_n = \alpha$.

(3) Let $\{(x_n, v_n)\}$ be a sequence such that (x_n, v_n) is a saddle point of (L_n, A, B); then $\{(x_n, v_n)\}$ has cluster points iff $\Omega_D \neq \phi$ and all the cluster points are saddle points of (L, A, B). Morever, in that case, the sequence $\{v_n\}$ actually converges and it's limit minimizes the function h on Ω_D.

PROOF : The existence of saddle value α follows from Theorem 3.1.

(1) Let $a_0 \in A$ be as in condition 2 of section 3, then $L(a_0, .)(1 - G_n)$ is upper compact. Therefore all the conditions of section 3 are satisfied for (L_n, A, B).

(2) Since α_n is the value of the problem $\{D_n\}$, Remark 4.1 and Theorem 2.1 imply that $\lim_{n \to \infty} \alpha_n = \alpha$.

(3) First we will prove that the sequence $\{x_n\}$ has cluster points and all its cluster points are solutions of $\{P\}$. Let $b \in B$, as in condition 4 of section 3. We have

$$L(x_n, b) = L_n(x_n, b)\,(1 - G_n(b))^{-1}.$$

By condition c, $1 - G_n(b)$ is invertible for each $n \in N$. Since (x_n, v_n) is a saddle point of (L_n, A, B), we have

$$L_n(x_n, b) \leqslant L_n(x_n, v_n) = \alpha_n.$$

Thus

$$L(x_n, b) \leqslant \alpha_n (1 - G_n(b))^{-1}.$$

The lower compactness of $L(., b)$ and the boundedness of $\alpha_n (1 - G_n(b))^{-1}$ imply that the sequence $\{x_n\}$ is included in a compact subset of A, and therefore has cluster points. Let u be an arbitrary element of B. We have

$$\begin{aligned} L(x_n, u) &= L_n(x_n, u)\ (1 - G_n(u))^{-1} \\ &\leqslant L_n(x_n, v_n)\,(1 - G_n(u))^{-1} \\ &= \alpha_n (1 - G_n(u))^{-1}. \end{aligned} \qquad \ldots(4.3)$$

Since $\alpha_n \to \alpha$ and $G_n(u)$ converges to zero, therefore for every $\epsilon > 0$, $\exists\ m \in N$ such that

$$L(x_n, u) \leqslant \alpha_n (1 - G_n(u))^{-1} \leqslant \frac{\alpha}{1 - \epsilon} \text{ for all } n > m$$

$$x_n \in T_{\alpha,\epsilon} = \left\{ x \in A : L(x, u) \leqslant \frac{\alpha}{1 - \epsilon} \right\} \text{ for all } n > m. \qquad \ldots(4.4)$$

Since $L(., u)$ is lower semicontinuous on A, therefore it follows that $T_{\alpha,\epsilon}$ is closed and any cluster point of $\{x_n\}$ belongs to $T_{\alpha,\epsilon}$. That is $L(\bar{x}, u) \leqslant \frac{\alpha}{1 - \epsilon}$; hence $L(\bar{x}, u) \leqslant \alpha$. Since $u \in B$ was arbitrary, therefore we conclude that $f(\bar{x}) = \sup_{u \in B} L(\bar{x}, u) \leqslant \alpha$. This is possible only if $f(\bar{x}) = \alpha$. Therefore $\bar{x} \in \Omega_P$. On the otherhand, Theorem 2.1, remark 2.2 and remark 4.1 imply that the sequence $\{v_n\}$ is convergent iff $\Omega_D \neq \phi$, and if $\bar{v} = \lim_{n \to \infty} v_n$ then $\bar{v} \in \Omega_D$ and $h(\bar{v}) \leqslant h(\omega)$ for all $\omega \in \Omega_D$.

5. Total One-Limit Modifications in Convex-Concave Saddle Point Problems

Let h: $E \times F \to \bar{R}$, where E and F are locally convex topological vector spaces. Let h satisfies the following :

(1) $h(., v)$ is strictly convex, lower semi-continuous, lower compact and bounded below on A for all $v \in B$.

(2) $h(x, .)$ is strictly concave, upper semi-continuous, upper compact and bounded above on B for all $x \in A$, where $A \subset E$ and $B \subset F$ are nonempty closed convex

sets. Consider a triplet (L, A, B) satisfying conditions of section 3 and the problem: $\{Z_n\}$ find the value and the saddle points of the triplet (L_n, A, B) where

$$L_n(x, v) = L(x, v)\left(1 + \frac{1}{\gamma_n} G(x, v)\right)$$

and $\{\gamma\}$ is a positive sequence such that $\lim_{n\to\infty} \gamma_n = +\infty$.

We can emulate the technique of Behar[1] to prove the following result analogous to Theorem 4.1 of Section 4.

Theorem 5.1—Under the assumptions 1-4 of Section 3 and 1,2 of Section 5 we have

(1) For all $n \in \bar{\mathbf{R}}$, the problem $\{Z_n\}$ has a unique saddle point $(x_n, v_n) \in A \times B$.

(2) The value $\alpha_n \to \alpha$, and the sequence $\{(x_n, v_n)\}$ converges to one of the saddle point of (L, A, B) which is the unique saddle point of the triplet (G, Ω_P, Ω_D) (where Ω_P and Ω_D are the solution sets of the primal and dual problems associated with the triplet (L, A, B).

Remark : We can give analogous treatment for translations in convex optimization, translations in Banach Space and partial translations in Banach Spaces with multiplicative type of modification si milar to $\inf\{f(x) H(x - z) : x \in A\}$ (see Behar[1]). We hope to give the details of such modifications in one of our future work.

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