[ndian J. pure appl. Math., 17(2) : 142-149 February 1986

THE LINE NEIGHBOURHOOD NUMBER OF A GRAPH*

E. SAMPATHKUMAR AND PRABHA S. NEERALAGI
Department of Mathematics, Karnatak University, Dharwad 580003

(*Received* 23 *January* 1985; *after revision* 8 *July* 1985)

For a point u and a line $x=uv$ in a graph $G = (V, E)$, let $N(u) = \{v \in V : uv \in E\}$ and $N[x] = N(u) \cup N(v)$. A set $T \subset E$ is a line-neighbourhood set (written ln-set) of G if $G = \bigcup_{x \in T} \langle N[x] \rangle$, where $\langle N[x] \rangle$ is the subgraph of G induced by $N[x]$. The line-neighburhood number $n'_0(G)$ of G is the minimum cardinality of an ln-set. Besides investigating some relationship of $n'_0(G)$ with other known parameters of G, many bounds for $n'_0(G)$ are obtained.

By a graph we mean a finite, undirected graph without loops and multiple lines. We follow the notation and terminology of Harary[4].

For a point u in a graph $G = (V, E)$ let $N(u) = \{v \in V : uv \in E\}$ and $N[u] = \{u\} \cup N(u)$. A set $S \subset V$ is a neighbourhood set (written n-set) of G if

$$G = \bigcup_{u \in S} < N[u] >$$

where $< N[u] >$ is the subgraph of G induced by $N[u]$.

The neighbourhood number $n_0(G)$ of G is the minimum cardinality of an n-set of G. This parameter was first introduced in Sampathkumar and Neeralagi[6]. The purpose of this paper is to introduce a new parameter called the line-neighbourhood number of G and investigate its relationship with other known parameters of G.

For a line $x = uv$ in G, let $N[x] = N(u) \cup N(v)$. A set T of lines of G is a line-neighbourhood set (written ln-set) of G, if $= G \bigcup_{x \in T} < N[x] >$.

The line-neighbourhood number $n'_0(G)$ of G is the minimum cardinality of an ln-set.

*This is a part of the second author's Ph.D thesis.

Throughout this paper we consider graphs without isolated points.

We first determine $n_0'(G)$ for some known graphs.

Proposition 1 — (a) $n_0' = 1$ for the following graphs.

(i) K_p, (ii) $K_p - x$, (iii) $K_{1,p-1}$, (iv) the wheel on p points.

(b) Suppose $x = uv$ is a line in G with $N(u) \cap N(v) = \phi$ and $\deg u + \deg v \geqslant p$. Then $n_0' = 1$. For example $n_0'(K_{m,n}) = 1$.

(c) If P_r is a path on r points,

$$n_0'(P_{3n-1}) = n_0'(P_{3n}) = n_0'(P_{3n+1}) = n$$

for $n = 1, 2, 3, \ldots$

(d) For a cycle C_n of length $n \geqslant 4$

$$n_0'(C_4) = 1, \quad n_0'(C_5) = n_0'(C_6) = 2$$

and

$$n_0'(C_{3r-2}) = n_0'(C_{3r-1}) = n_0'(C_{3r}) = r$$

for $r = 3, 4, 5, \ldots$

For a graph G, let $\alpha_0(G)$ and $\alpha_1(G)$ be the point and line covering numbers of G; $\beta_0(G)$ and $\beta_1(G)$ be the point and line independence numbers of G respectively.

A set $D \subset V$ is a dominating set of G if every point in $V - D$ is adjacent to some point in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A detailed study of $\gamma(G)$ may be found in Cockayne and Hedetniemi[2] and Walikar *et al.*[7]

A set $T \subset E$ is a line dominating set of G if every line in $E - T$ is adjacent to some line in T. The line domination number $\gamma'(G)$ of G is the minimum cardinality of a line dominating set[3].

Let G be a (p, q) graph with maximum degree Δ and chromatic number $\chi(G)$.

For the subgraph $\langle S \rangle$ of G induced by a set $S \subset V$, let $q(\langle S \rangle)$ be

the number of lines in $\langle S \rangle$, and $q_0 = \min \{ q \langle D \rangle : D$ is a minimum dominating set of $G\}$.

The following known results will be used in the sequel.

Proposition A (Sampathkumar and Neeralagi[6])—(a) For any graph G

$$\gamma(G) \leqslant n_0(G) \leqslant \alpha_0(G) \qquad \ldots(1)$$

$$n_0(G) \leqslant \alpha_1(G) \qquad \ldots(2)$$

$$\gamma(G) \leqslant n_0(G) \leqslant (\chi(G) - 1)\,\beta_0(G), \text{ where } \chi(G) \geqslant 2. \qquad \ldots(3)$$

(b) For a bipartite graph G which is not totally disconnected,

$$\gamma(G) \leqslant n_0(G) \leqslant \beta_0(G) \leqslant \chi(\bar{G}) \qquad \ldots(4)$$

where $\bar{G}$ is the complement of G.

(c) For any (p, q) graph G,

$$p - q + q_0 \leqslant n_0(G) \leqslant p - \Delta \qquad \ldots(5)$$

$$\left\{ \frac{p}{\Delta + 1} \right\} \leqslant n_0(G) \leqslant p - \beta_0 \qquad \ldots(6)$$

We now prove :

Proposition 2—For a graph G

$$\frac{1}{2}\gamma \leqslant n_0' \leqslant n_0 \qquad \ldots(7)$$

$$n_0' \leqslant \gamma' \leqslant \min\{\alpha_0, \alpha_1, \beta_1\} \qquad \ldots(8)$$

$$\frac{1}{2}\gamma \leqslant n_0' \leqslant n_0 \leqslant \min\{\alpha_0, \alpha_1\}. \qquad \ldots(9)$$

PROOF : Let $T = \{x_1, x_2, \ldots x_r\}$ be a minimum ln-set of G and $x_i = u_i v_i$, $1 \leqslant i \leqslant r$. Let $H = \bigcup_{i=1}^{r} \{u_i, v_i\}$. Clearly H is a dominating set of G and $\gamma \leqslant |H| \leqslant 2|T| = 2n_0'$. Thus $\frac{1}{2}\gamma \leqslant n_0'$. To prove the other inequality in (7), let $S = \{v_1, v_2, \ldots v_r\}$ be a minimum n-set of G. Choose one line y_i that joins v_i with some point

in $V(G) - S$. This is possible because we observe that if S is a minimum n-set, then $V(G) - S$ is a dominating set. Now $M = \{y_1, y_2, \ldots y_r\}$ is an ln-set of G and hence

$$n_0' \leqslant |M| = n_0.$$

This proves (7).

To establish (8), we observe that the first half of (8) follows from the fact that every line dominating set is an ln-set. To prove the second half, let $\{v_1, v_2, \ldots v_k\}$ be a minimum point cover and x_i be a line incident to v_i. Clearly $\{x_1, x_2, \ldots x_k\}$ is a line dominating set. So $\gamma' \leqslant \alpha_0$. Also any line cover of G, as well as any maximum independent set of lines are line dominating sets of G. Thus $\gamma' \leqslant \alpha_1$ and $\gamma' \leqslant \beta_1$, and $\gamma' \leqslant \min \{\alpha_0, \alpha_1, \beta_1\}$.

This proves (8). Now (9) is a consequence of (1), (2) and (7).

Note : There is no relation between γ' and n_0. For the cycle C_5, $n_0 = 3$, $\gamma' = 2$, and for K_4, $n_0 = 1$ and $\gamma' = 2$. Also it may happen that $\gamma < n_0'$ or $\gamma > n_0'$. For G in Fig. 1, $\gamma = 2$ and $n_0' = 3$ and for the path P_4, $n_0' = 1$, $\gamma = 2$.

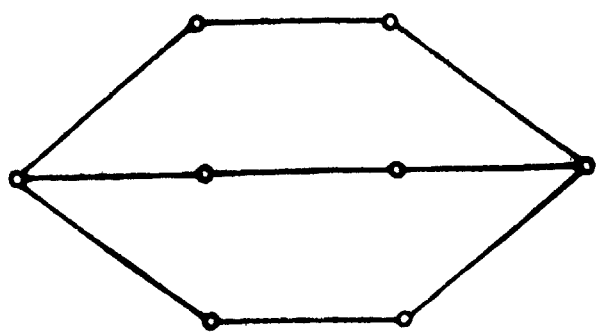

FIG. 1.

For a real number r, let $[r]$ be the greatest integer not exceeding r, and $\{r\} = -[-r]$.

Proposition 3—For a graph G of order p,

$$n_0' \leqslant \gamma' \leqslant \frac{p}{2} \quad . \qquad \ldots(10)$$

PROOF : Since $\alpha_1 + \beta_1 = p$ (see Harary[4], p. 95), we have by (8)

$$n_0' \leqslant \gamma' \leqslant \frac{\alpha_1 + \beta_1}{2} = \frac{p}{2}.$$

Proposition 4—(a) For any graph G with $\chi(G) \geq 2$

$$n_0'(G) \leq (\chi(G) - 1)\,\beta_0(G). \quad \ldots(11)$$

(b) For a bipartite graph G which is not totally disconnected

$$n_0'(G) \leq \beta_0(G) \leq \chi(\bar{G}). \quad \ldots(12)$$

PROOF : (11) is a consequence of (3) and (7) and (12) is a consequences of (4) and (7).

The following result appears in Waliker *et al.*[7].

Proposition B (Walikar *et al.*[7])—For a (p, q) graph G,

$$p - q + q_0 \leq \gamma(G) \leq p - \Delta \quad \ldots(13)$$

$$\left\{ \frac{p}{\Delta + 1} \right\} \leq \gamma(G) \leq p - \beta_0 + p_0 \quad \ldots(14)$$

where p_0 is the number of isolated points in G.

Proposition 5— For a (p, q) graph G

$$\frac{1}{2}(p - q + q_0) \leq n_0'(G) \leq p - \Delta \quad \ldots(15)$$

$$\frac{1}{2}\left\{ \frac{p}{\Delta + 1} \right\} \leq n_0'(G) \leq p - \beta_0. \quad \ldots(16)$$

PROOF : The lower bounds in (15) and (16) follow from (7), (13) and (14), where as the upper bound follow from (5), (6) and (7).

The following result appears in Jaegar and Payan[5].

Proposition C—For any graph G of order p

$$\min\{3, p + 1\} \leq \gamma(G) + \gamma(\bar{G}) \leq p + 1 \quad \ldots(17)$$

$$\min\{2, p\} \leq \gamma(G)\,\gamma(\bar{G}) \leq p. \quad \ldots(18)$$

Proposition 6—Let G be a graph of order p. For $t = n_0'$ or γ', we have

$$\frac{1}{2}\min\{3, p + 1\} \leq t(G) + t(\bar{G}) \leq 2\left[\frac{p}{2}\right]. \quad \ldots(19)$$

$$\frac{1}{4}\min\{2, p\} \leqslant t(G)\, t(\bar{G}) \leqslant \left[\frac{p}{2}\right]^2. \qquad \ldots(20)$$

PROOF : The lower bounds follow from (7), (10), (17) and (18). To establish the upper bounds, we have $t(G) \leqslant \left[\frac{p}{2}\right]$ by Propostion 3. Hence the upper bounds in (19) and (20) follow.

To obtain some more bounds, we first state some known results. Let $L(G)$ be the line graph of G.

Proposition D (Gupta[3])—For a (p, q) graph G,

$$\beta_0(L(G)) = \beta_1(G) \qquad \ldots(21)$$

$$\alpha_0(L(G)) = q - \beta_1(G) \qquad \ldots(22)$$

$$\gamma(L(G)) = \gamma'(G) \qquad \ldots(23)$$

$$\alpha_1(L(G)) = \left\{\frac{q}{2}\right\}. \qquad \ldots(24)$$

We now prove,

Proposition 7—Let G be a graph such that $L(G)$ has no isolated points, then

$$n_0'(G) \leqslant \gamma'(G) \leqslant n_0(L(G)). \qquad \ldots(25)$$

PROOF : The result follows from (8), (1) and (23).

Corollary 7.1—For any graph G with q lines.

$$\text{(i) } \gamma' \leqslant \left\{\frac{q}{2}\right\} \text{ and (ii) } \gamma' \leqslant q - \beta_1(G).$$

PROOF : (i) By (2), $n_0(L(G)) \leqslant \alpha_1(L(G)) = \left\{\frac{q}{2}\right\}$

because of (24). Now (i) follows from (25).

(ii) By (25), (1) and (22),

$$\gamma'(G) \leqslant n_0(L(G)) \leqslant \alpha_0(L(G)) = q - \beta_1(G).$$

Note that the bounds in (i) and (ii) are also bounds for n_0'.

To obtain some bounds for $n_0'(G)$ involving connectivity $k(G)$ and line connectivity $\lambda(G)$ of G, we state some known results.

Proposition E (Walikar *et al.*[7])—For any graph G of order $p \geq 2$,

$$\gamma(G) \leq p - k(G). \qquad \ldots(26)$$

Further, for a connected graph G, if the equality holds in (26), then G is regular.

Proposition F (Chartrand and Stewart[1])—If G is a graph for which $\lambda(G) \geq 2$, then

$$\lambda(G) \leq k(L(G)). \qquad \ldots(27)$$

We now prove,

Proposition 8— (a) If G is a graph with $q \geq 2$ lines and $\lambda(G) \geq 2$, then

$$n_0'(G) \leq \gamma'(G) \leq q - \lambda(G). \qquad \ldots(28)$$

(b) If $\gamma'(G) = q - k(I(G))$, then $L(G)$ is regular, if $L(G)$ is connected.

PROOF : (a) Since $L(G)$ has q points, we have by (23) and (26),

$$\gamma'(G) = \gamma(L(G)) \leq q - k(L(G))$$
$$\leq q - \lambda(G)$$

because of (27). (b) This follows from Proposition E.

Note : This converse of (b) is not true, for example consider the cycle C_6.
We now characterize graphs for which (i) $n_0' = \gamma'$ and (ii) $n_0' = \gamma' = \alpha_1$.

Proposition 9—For a graph G, $n_0' = \gamma'$ if and only if there exists a minimum ln-set T which is a line dominating set.

PROOF : Let $n_0' = \gamma'$ and T be a minimum line dominating set. Then T is a ln-set and since $n_0' = \gamma'$, T is a minimum ln-set. Conversely, the given condition implies $\gamma' \leq |T| = n_0'$. Since $n_0' \leq \gamma'$ by (8) we have $n_0' = \gamma'$.

Corollary 9.1—If G has no cycles of length ≤ 4, then $\gamma' = n_0'$.

PROOF : For a line x in G, let G_x be the subgraph of G containing precisely x and all lines adjacent to x. Since G has no cycles of length $\leqslant 4$, $G_x = \langle N[x] \rangle$. If T is any minimum ln-set of G, then since

$$G = \bigcup_{x \in T} \langle N[x] \rangle = \bigcup_{x \in T} G_x$$

it follows that T is a line dominating set. Hence $n'_0 = \gamma'$ by Proposition 9.

Corollary 9.2—If G is a bipartite graph without quadrilaterals, then $n'_0 = \gamma'$. In particular, for any tree, $n'_0 = \gamma'$.

Proposition 10— For a graph G, $n'_0 = \alpha_1$ if and only if there exists a minimum ln-set T of G which is also a line cover of G.

The Proof is similar to that of Proposition 9.

Note : In view of (8) and Proposition 10, we have $n'_6 = \gamma' = \alpha_1$ for any graph with $n'_0 = \alpha_1$.

The following Corollary is immediate.

Corollary 10.1—For a graph G, (i) $n'_0 = \alpha_1$ if and only if there exists a minimum ln-set T such that every point of G is incident to some line in T.

(ii) $n' < \alpha_1$ if and only if for every minimum ln-set T there exists a point not incident to any line in T.

ACKNOWLEDGEMENT

Thanks are due to the referee for his comments.

REFERENCES

1. G. Chartrand, and M. J. Stewart, *Math. Ann.* **182** (1969), 170–74.
2. E. J. Cockayne, and S. T. Hedetniemi, *Networks* **7** (1977), 247–61.
3. R. P. Gupta, Independence and covering numbers of Line graphs and Total graphs, *Proof Techniques in Graph Theory*, (ed. : F. Harary). Academic Press, New York, 1969.
4. F. Harary, *Graph Theory*. Addison Wesley, Reading, Mass, (1969).
5. F. Jaegar, and C. Payan, *C R. Acad. Sc. Paris Ser A*, t-274 (1972), 729–30.
9. E. Sampathkumar, and Prabha S. Neeralagi, *Indian J. pure & appl. Math.* **16** (1985), 126–32.
7. H. B. Walikar, B. D. Acharya, and E. Sampathkumar, *Recent Developments in the Theory of Domination in Graphs*, M. R. I. Lecture Notes No. 1, The Metha Research Institute of Mathematies and Mathematical Physics Allahabad, 1979.