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A VARIABLE DROP THEOREM AND A FIXED POINT THEOREM VIA MAXIMAL ELEMENT TECHNIQUES

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We generalize a drop theorem of Turinici[5] via maximal element technique. We also give a generalization of a fixed point theorem of Aubin and Siegel[1] as an application of the same technique.

1. INTRODUCTION

Let (X, d) be a metric space and let $\leqq$ be an ordering on X (i.e., a reflexive, antisymmetric and transitive relation on X). For every element $x \in X$ and every (nonempty) subset $Y \subset X$, let $Y(x, \leqq)$ denote the subset of all $y \in Y$ with $x \leqslant y$. An element $z \in X$ is said to be an upper bound of $Y \subset X$ iff $z \in X(y, \leqq)$, for all $y \in Y$, and a maximal element of Y iff $\{z\} = Y(z, \leqq)$. Also, a subset C of X is called a chain in X iff, for every $x, y \in C$. either $x \leqslant y$ or $y \leqslant x$. The ordering $\leqq$ on X considered is said to be semiclosed iff $X(x, \leqq)$ is a closed subset of X for every $x \in X$. $T : X \to X$ is said to be a closed mapping iff it satisfies $x_n \to x$, $Tx_n \to x'$ imply $Tx = x'$.

A Maximal Element Result :

Following theorem is proved in Turinici[5]. Let (X, d) be a metric space.

Theorem 1.1—Suppose the ordering $\leqq$ on X and the function Φ from X into R_+ are such that

(i) $\leqq$ is a semi-closed ordering

(ii) for every couple $x, y \in X$ with $x \leqslant y$ we have the relation

$$d(x, y) \leqslant \Phi(x) - \Phi(y).$$

Then for every $x \in X$, there is a maximal element $z \in X$ with $x \leqslant z$. Define $D(x, y) = \max \{d(x, y), cd(Tx, Ty)\}$ for some $c > 0$. Since T is a closed mapping, It is easy to show that D is a metric on X.

We generalize this result in the following way :

Theorem 1.2—Suppose $T: X \to X$ is a closed mapping. Suppose the ordering $\leqq$ on X and the function Φ from X into R_+ are such that

(i) $\leqq$ is a semi-closed ordering.

(ii) For every couple $x, y \in X$ with $x \leqslant y$ we have the relation

$$\max \{d(x, y), cd(Tx, Ty)\} \leqslant \Phi(Tx) - \Phi(Ty)$$

for some $c > 0$. Then for every $x \in X$, there is a maximal element $z \in X$ with $x \leqslant z$.

Remark 1 : To see that Theorem 1.2 is indeed an extension of Theorem 1.1 we need to put $T = I$ (identity may) and $c = 1$ in Theorem 1.2 to get Theorem 1.1.

Remark 2 : The relation (ii) in Theorem 1.2 was first considered by Park[4] (Theorem 1, p. 145).

Proof of Theorem 1.2—We let $D(x, y) = \max \{d(x, y), cd(Tx, Ty)\}$ for $x, y \in X$. As observed above D is a metric on X. We show that $\leqq$ is semi-closed with respect to D. Let $y_n \in X(x, \leqq)$ and $D(y_n, x') \to 0$. Also we have $d(y_n, x') \leqslant D(y_n, x')$, thus $d(y_n, x') \to 0$. But $\leqq$ is semi-closed with respect to d, hence $x' \in X(x, \leqq)$, i.e., $X(x, \leqq)$ is closed (with respect to D) for each $x \in X$. Next we have $D(x, y) \leqslant \Phi(Tx) - \Phi(Ty)$ for $x, y \in X$ with $x \leqslant y$. Hence by replacing Φ by the composition map $\Phi \circ T$ we see that both the conditions (i) and (ii) of Theorem 1.1 are satisfied. Hence the result follows from Theorem 1.1.

As an application of this result we give an extension of a drop theorem of Turinici[5]. We give some notational conventions and preliminary facts before giving our result. Let $(X, \|.\|)$ be a Banach Space. For every $x, y \in X$, let $[x, y]$ denote the linear segment between x and y (the subset of all $z \in X$ with $z = tx + (1 - t)y$ for some $0 \leqslant t \leqslant 1$); also, for any (non-empty) bounded subset $A \subset X$, let diam (A) and $bd(A)$ denote the diameter and the boundary of A respectively, and moreover, for every $x \in X$, let Dist (x, A) (dist (x, A)) denote the supremum (infimum) of all distances $\|x - a\|$, $a \in A$. Suppose C is a (nonempty) bounded closed convex subset of X and let x ($x \in X$) does not belong to C. By a drop $[x, C]$ generated by x and C we mean convex hull of $\{x\} \cup C$ (Danes[3], Brϕndsted)[2], the segment notation is motivated by the fact that, evidently, $[x, C]$ is the union of all segments $[x, c]$ with $c \in C$. Clearly every drop is also a (nonempty) bounded closed convex subset of X. In what

follows, C is a (nonempty) bounded closed convex subset of X and Y a (nonempty) subset of X satisfying.

(*) Y is closed and disjoint from C.

In this case, define a relation $\leqq$ on Y by

$$x, y \in Y, x \leqslant y \text{ iff } y \in [x, C]. \qquad \ldots(1.1)$$

One can refer to Turinici[5] to see that $\leqq$ defines an ordering on Y. Let $T : X \to X$ be a closed linear mapping. Then it is easily seen that for $x \in X$, and $c > 0$

$$|||x||| = \max \{\|x\|, c\|Tx\|\}$$

defines a norm on X and $(X, |||\,.\,|||)$ is a Banach Space under the new norm. Define $\text{Dist}_N(., C)$ and $\text{dist}_N(., C)$ under the induced norm accordingly. Suppose also that the following hypothesis is fulfilled (**) $\delta = \text{diam}\ (C) < \rho = \inf \{\text{Dist}_N\ (x, C) : x \in Y\}$. Then we have the following lemma which is an extension of Lemma 1 of Turinici[5].

Lemma 1.3—Under the above conventions, denote

$$f = \text{Dist}_N(., C), \alpha = \frac{\delta}{\rho}, \beta = \frac{1}{1-\alpha}; \text{ then}$$

$$x, y \in Y, y \in [x, C] \text{ imply } |||x - y||| \leqslant \beta (f(x) - f(y)). \qquad \ldots(1.2)$$

PROOF: Let $x, y \in Y$ be such that $y \in [x, C]$; then $y = tx + (1 - t)c$ for some $0 \leqslant t \leqslant 1$, $c \in C$. Let $a \in C$ be arbitrary and fixed. From the evident relations

$x - y = (1 - t)(x - c)$, $y - a = t(x - a) + (1 - t)(c - a)$ we get immediately the inequalities

$$|||x - y||| + \beta |||y - a||| \leqslant (1 - t) |||x - c||| + \beta t |||x - a||| + \beta (1 - t) |||c - a|||$$

and on the other hand by (**),

$$|||c - a||| \leqslant \text{diam}\ (C) \leqslant \alpha \inf \{f(x) : x \in Y\} \leqslant \alpha f(x).$$

So that, by combining these relations,

$$|||x - y||| + \beta |||y - a||| \leqslant (1 - t) |||x - c||| + \beta t |||x - a|||$$

(equation continued on p. 178)

$$+ \beta\alpha(1-t)f(x) \leqslant (1 - t + \beta t + \beta\alpha(1-t))f(x)$$

$$= ((1 + \beta\alpha)(1-t) + \beta t)f(x) = \beta f(x)$$

also since $|||y - a||| \geq f(y)$, hence,

$$|||x - y||| \leqslant \beta f(x) - \beta |||y - a||| \leqslant \beta(f(x) - f(y))$$

which completes the proof of the lemma.

Assumption A—The following result is true for closed mapping $T : X \to X$ which satisfy the assumption (as inherent in Lemma 1.3) $|||x - y||| \leqslant \beta[f(T(x)) - f(T(y))]$, for all $x, y \in Y$ and $y \in [x, C]$.

Theorem 1.4—Let $(X, |||.|||)$ be the Banach Space with C and Y such that (*) and (**) are satisfied. Then for every element $x \in Y$, there exists $z \in Y$ satisfying (a) $z \in [x, C]$, (b) $\{z, C\} \cap Y = \{z\}$, (c) $z \in bd(Y)$, provided assumption A is satisfied.

PROOF : We define the ordering $\leqq$ by (1). To show that $\leqq$ is a semi-closed ordering we see that if $Y(x, \leqq)$ is such that $x \leqslant y_n$ with $y_n \in [x, C]$ and $y_n \to y$, then $y \in [x, C]$, since every drop is also a (nonempty) bounded closed convex subset of X. From which it follows that $Y(x, \leqq)$ is a closed subset of Y (via the relation (1) again) for each x. Hence the hypothesis (i) of Theorem 1.2 holds. On the other hand invoking the assumption A we see that (ii) holds with $\Phi = \beta(f \circ T)$. Consequently Theorem 1.2 applies and since $\leqq$ is defined via relation (1), it is easy to observe that the conclusion of Theorem 1.2 implies that the conditions (a), (b) and (c) hold.

Remark : The assumption A gives some sort of growth condition on the function $f \circ T$.

2. A FIXED POINT RESULT FOR A MULTIVALUED MAPPING

Aubin and Siegel[1] gave a fixed point theorem for a multivalued mapping. We generalize this result in this section. For notations and preliminary facts we proceed as follows :

$A : X \to 2^X$ is a multivalued map.

Definition 2.1—A function $\Psi : X \to R_+$ (X a complete metric space) is called a weak entropy of A if for all $x \in X$, there exists $y \in A(x)$ such that

$$d(x, y) \leqslant \Psi(x) - \Psi(y). \qquad \ldots(2.1)$$

A map $A : X \to 2^X$ is called weakly dissipative if there exists a weak entropy of A.

Definition 2.2—A sequence $\{x_n\}_{n\geq 0}$ of elements is in X is called a trajectory starting at x if $x_0 = x$ and $x_{n+1} \in A(x_n)$, $n \geq 0$. We denote the set of all such trajectories by $\mathcal{T}(A, x)$.

Theorem 2.1—(Aubin and Siegel[1])—Any weakly dissipative closed map $A : X \to 2^X$ has a fixed point; furthermore for any $x_0 \in X$, there exists a trajectory starting at x_0 and converging to a fixed point.

We generalize the concept of weak entropy in the following way.

Definition 2.3—Let $T : X \to X$ be a closed mapping. Then a function $\Psi : X \to R_+$ is called a weak T-entropy of A if for all $x \in X$, there exists $y \in A(x)$ such that

$$\max \{d(x, y), cd(Tx, Ty)\} \leqslant \Psi(Tx) - \Psi(Ty) \qquad \ldots(2.2)$$

A map $A : X \to 2^X$ is said to be weakly T-dissipative if there exists a weak T-entropy of A. Then we have the following extension of the Theorem 2.1.

Theorem 2.2—Any weakly T-dissipative closed map A has a fixed point. Let $x \in X$, then the sequence defined by $x_{n+1} \in A(x_n)$ converges to a fixed point of A.

PROOF : Let $D(x, y) = \max \{d(x, y), cd(Tx, Ty)\}$. Since T is a closed mapping, it is easy to show that D is a metric on X. Next we show that with metric D if A is closed then it is closed with respect to d also. For simplicity we take the range of A as a singleton set. Then it is not difficult to emulate the proof for a set valued map A by introducing Hausdorff metric. Let $D(x_n, x') \to 0$ and $D(Ax_n, y) \to 0$, then since A is closed $D(Ax', y) = 0$. But $d(x_n, x') \leqslant D(x_n, x')$ and $d(Ax_n, y) \leqslant D(Ax_n, y)$, hence $d(x_n, x') \to 0$ and $d(Ax_n, y) \to 0$ simultaneously. But $D(Ax', y) = 0$ and $d(Ax', y) \leqslant D(Ax', y)$, from which it follows that $d(Ax', y) = 0$, hence A is closed with respect to d. Now setting $\Psi = \Psi \circ T$ in Theorem 2.2 we see that A satisfies the hypothesis of Theorem 2.1 and thus we get the proof of Theorem 2.2 completed by the conclusion of Theorem 2.1.

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