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SOME THEOREMS ON FIXED POINTS IN PSEUDO COMPACT TICHONOV SPACE

H. K. PATHAK
Department of Mathematics, Kalyan Mahavidyalaya
Bhilainagar (M. P.) 490006

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In the present paper we established some fixed point theorems in pseudocompact Tichonov space. These theorems include as a special case the fixed point theorems recently established by Jain and Dixit[4], Harinath[3] and also include some interesting results of Fisher[2] over compact metric spaces.

INTRODUCTION

Let X be a topological space, X is said to be pseudocompact if every real valued continuous function on X is bounded. It is known that every compact space is pseudocompact, but the converse is not true (Englaking[1], Ex. 5, p. 150). However, in a metric space the notions: 'compact' and 'pseudocompact' coincide. By Tichonov space we mean a completely regular Hausdorff space. It may be observed that the product of two Tichonov spaces is again a Tichonov space whereas the product of two pseudocompact spaces need not be pseudocompact.

MAIN RESULTS

First of all, we establish the following :

Theorem 1—Let P be a pseudocompact Tichonov space and μ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonovb ut need not be pseudocompact). Suppose μ also satisfies

$$\text{(i)} \begin{cases} \mu(x, x) = 0 \text{ for all } x \in P \text{ and} \\ \mu(x, z) \leqslant \mu(x, z) + \mu(z, y) \text{ for all } x, y, z \in P. \end{cases}$$

If S and T are two continuous self maps of P satisfying

(ii) $ST = TS$; and

$$\text{(iii)} \quad \mu(STx, Sy) < \alpha_1 \mu(Tx, y) + \alpha_2 \mu(STx, Tx) + \alpha_3 \mu(STx, y) + \alpha_4 \mu(Tx, Sy) + \alpha_5 \mu(Sy, y) + \alpha_6 \frac{\mu(STx, Tx)\, \mu(Sy, y)}{\mu(Tx, y)} + \alpha_7 \frac{\mu(Tx, Sy)\, \mu(STx, y)}{\mu(Tx, y)}$$

for all distinct $x, y \in P$ with $Tx \neq y$, where $\alpha_i \geqslant 0$, $\alpha_2 + \alpha_3 + \alpha_6 < 1$, $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 \leqslant 1$.

Then S and T have a common fixed point in P which is unique whenever $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1$

PROOF : We define $\phi : P \to R$ by $\phi(p) = \mu(STp, Tp)$ for all $p \in P$, where R is the set of real numbers. Clearly ϕ is continuous being the composite of three continuous functions S, T and μ. Since P is pseudocompact Tichonov space, every real valued continuous function over P is bounded and attain its bounds. Thus there exists a point $\nu \in P$ such that $\phi(\nu) = \inf[\phi(p)/p \in P]$, where 'inf' denotes the infimum or the greatest lower bound in R. It may be noted that $\phi(p) \subset R$. We now affirm that ν is a fixed point for S. If not, let us suppose that $S\nu \neq \nu$. Then using (ii) and (iii), we have

$$\begin{aligned} \phi(S\nu) &= \mu(STS\nu, TS\nu) = \mu(STS\nu, ST\nu) \\ &< \alpha_1 \mu(TS\nu, T\nu) + \alpha_2 \mu(STS\nu, TS\nu) + \alpha_3 \mu(STS\nu, T\nu) \\ &\quad + \alpha_4 \mu(TS\nu, ST\nu) + \alpha_5 \mu(ST\nu, T\nu) \\ &\quad + \alpha_6 \frac{\mu(STS\nu, TS\nu)\, \mu(ST\nu, T\nu)}{\mu(TS\nu, T\nu)} \\ &\quad + \alpha_7 \frac{\mu(TS\nu, ST\nu)\, \mu(STS\nu, T\nu)}{\mu(TS\nu, T\nu)} \end{aligned}$$

or $\quad (1 - \alpha_2 - \alpha_3 - \alpha_6)\, \phi(S\nu) < (\alpha_1 + {}_3 + \alpha_5)\, \phi(\nu) \quad (\because \alpha_3 \geqslant 0)$

or $\quad \phi(S\nu) < \phi(\nu) \quad (\because \alpha_2 + \alpha_3 + \alpha_6 < 1,$

$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 \leqslant 1)$

a contradiction. Hence $\nu \in P$ is a fixed point for S i. e. $S\nu = \nu$.

Using (ii), we have

$$ST\nu = TS\nu = T\nu. \qquad \ldots(*)$$

Now we shall prove that $T\nu = \nu$. If possible, let $T\nu \neq \nu$. Then, we have by (*) and (iii)

$$\mu(T\nu, \nu) = \mu(ST\nu, S\nu)$$

$$< \alpha_1 \mu(T\nu, \nu) + \alpha_2 \mu(ST\nu, T\nu) + \alpha_3 \mu(ST\nu, \nu)$$

$$+ \alpha_4 \mu(T\nu, S\nu) + \alpha_5 \mu(S\nu, \nu)$$

$$+ \alpha_6 \frac{\mu(ST\nu, T\nu)\, \mu(S\nu, \nu)}{\mu(T\nu, \nu)} + \alpha_7 \frac{\mu(T\nu, S\nu)\, \mu(ST\nu, \nu)}{\mu(T\nu, \nu)}$$

$$< (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7)\, \mu(T\nu, \nu)$$

$$< \mu(T\nu, \nu) \qquad (\because \alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1)$$

leading to a contradiction and hence $\nu \in P$ is a fixed point of T i. e. $T\nu = \nu$.

To prove the uniqueness of ν, if possible, let ω be another fixed point for S and T i. e. $\nu = S\nu = T\nu$ and $\omega = S\omega = T\omega$ $(\omega \neq \nu)$.

Then, using (iii), we get

$$\mu(\nu, \omega) = \mu(ST\nu, S\omega)$$

$$< (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7)\, \mu(\nu, \omega)$$

$$< \mu(\nu, \omega) \qquad (\because \alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1)$$

again leading to a contradiction which proves that $\nu \in P$ is unique. This completes the proof.

If we put $T = S$ in Theorem 1, we obtain the following :

Corollary—Let P be a pseudocompact Tichonov space and μ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but not need not be Pseudocompact). Suppose μ also satisfies

$$\text{(i)} \begin{cases} \mu(x, x) = 0 \text{ for all } x \in P \text{ and} \\ \mu(x, y) \leqslant \mu(x, z) + \mu(z, y) \text{ for all } x, y, z \in P. \end{cases}$$

If $S : P \to P$ is a continuous map satisfying

$$\text{(ii)} \quad \mu(S^2x, Sy) < \alpha_1 \mu(Sx, y) + \alpha_2 \mu(S^2x, Sx) + \alpha_3 \mu(S^2x, y)$$

$$+ \alpha_4 \mu(Sx, Sy) + \alpha_5 \mu(Sy, y)$$

(equation continued on p. 183)

$$+ \alpha_6 \frac{\mu(S^2x, Sx)\,\mu(Sy, y)}{\mu(Sx, y)}$$

$$+ \alpha_7 \frac{\mu(Sx, Sy)\,\mu(S^2x, y)}{\mu(Sx, y)}$$

for all distinct $x, y \in P$ with $Sx \neq y$ where $\alpha_3 \geqslant 0$,

$$\alpha_2 + \alpha_3 + \alpha_6 < 1,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 \leqslant 1.$$

Then S has a fixed point in P, which is unique, whenever

$$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1.$$

If we omit condition (i) and take $\alpha_3 = \alpha_4 = \alpha_7 = 0$ in Theorem 1, we obtain the following :

Theorem 2—Let P be a pseudocompact Tichonov space and μ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but need not be pesudocompact). If S and T are two continuous self maps of P satisfying

(a) $ST = TS$; and

(b) $\mu(STx, Sy) < \alpha_1 \mu(Tx, y) + \alpha_2 \mu(STx, Tx)$

$$+ \alpha_5 \mu(Sy, y) + \alpha_6 \frac{\mu(STx, Tx)\,\mu(Sy, y)}{\mu(Tx, y)}$$

for all distinct $x, y \in P$ with $Tx \neq y$, where

$$\alpha_2 + \alpha_6 < 1,\quad \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 \leqslant 1$$

then S and T have a common fixed point in P.

Every metric space is a Hausdorff space. Hence as a particular case of Theorem 1, we have the following result on a compact metric space.

Theorem 3—Let (M, d) be a compact metric space and S and T are two continuous self maps of M satisfying

(a) $ST = TS$; and

(b) $d(STx, Sy) < \alpha_1 d(Tx, y) + \alpha_2 d(STx, Tx) + \alpha_3 d(STx, y)$

$$+ \alpha_4 d(Tx, Sy) + \alpha_5 d(Sy, y)$$

$$+ \alpha_6 \frac{d(STx, Tx)\,d(Sy, y)}{d(Tx, y)}$$

(*equation continued on p.* 184)

$$+ \alpha_7 \frac{d(Tx, Sy)\, d(STx, y)}{d(Tx, y)}$$

for all distinct $x, y \in M$ with $Tx \neq y$, where $\alpha_3 \geqslant 0$,

$$\alpha_2 + \alpha_3 + \alpha_6 < 1,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 \leqslant 1$$

and

$$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1$$

then S and T have a unique common fixed point in M. If we put $T = S$ in Theorem 3, we get the following :

Corollary—Let (M, d) be a compact metric space and $S : M \to M$ be a continuous map satisfying

$$d(S^2x, Sy) < \alpha_1 d(Sx, y) + \alpha_2 d(S^2x, Sx) + \alpha_3 d(S^2x, y)$$

$$+ \alpha_4 d(Sx, Sy) + \alpha_5 d(Sy, y)$$

$$+ \alpha_6 \frac{d(S^2x, Sx)\, d(Sy, y)}{d(Sx, y)}$$

$$+ \alpha_7 \frac{d(Sx, Sy)\, d(S^2x, y)}{d(Sx, y)}$$

for all distinct $x, y \in M$ with $Sx \neq y$, where $\alpha_3 \geqslant 0$,

$$\alpha_2 + \alpha_3 + \alpha_6 < 1,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 \leqslant 1$$

and

$$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leqslant 1$$

then S has a unique fixed point in M.

Finally, we furnish examples to discuss the validity of the hypothesis and degree of generality of Theorem 1.

Example 1—Let $P = \{0, 1, 2, 3\}$ and let $\mathscr{D}$ is the discrete topology on P. Suppose S and T are continuous self maps of P defined by $S0 = 0$, $S1 = 2$, $S2 = 3$.

$S3 = 1$, $T0 = 0$, $T1 = 3$, $T2 = 1$, $T3 = 2$

and suppose μ be a non-negative real valued continuous function over $P \times P$ such that

$$\mu(x, y) = x - y, \text{ for all distinct } x, y \in P.$$

Then, it is clear that P is a pseudocompact Tichonov space and S and T satisfy all the conditions of Theorem 1.

Now setting $x = 1$, $y = 2$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{12}$,

$\alpha_4 = \frac{1}{2}$, $\alpha_5 = 0$ $\alpha_6 = \frac{1}{2}$, $\alpha_7 = \frac{1}{2}$,

we observe that S and T satisfy condition (iii) and obviously 0 is the unique common fixed point of S and T.

Example 2—Let $P = [0, 1]$ be a metric space with usual metric μ in which the topology is induced by metric μ.

Suppose S and T are continuous self maps of P defined by

$$Sx = \frac{x}{2}, \text{ for all } x \in [0, 1]$$

$$Tx = \frac{x}{3}, \text{ for all } x \in [0, 1].$$

Since every metric space is Tichonov space and also the notions: 'compact' and pseudo-compact' coincide in a metric space, the space P is a pseudocompact Tichonov space.

Also, it is clear that S and T satisfy all the conditions of Theorem 1.

Now setting $x = 0$, $y = 1$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{6}$, $\alpha_4 = \frac{1}{6}$, $\alpha_5 = \frac{1}{6}$, $\alpha_6 = \frac{1}{6}$, $\alpha_7 = \frac{2}{3}$, we observe that S and T satisfy condition (iii) and obviously 0 is the unique common fixed point of S and T.

Remarks : (1) On taking $T = I_P$ (identity map of P) in Theorems 1, 2 and 3 we obtain Theorems 1, 2 and 3 of Jain and Dixit[4].

(2) On taking $T = I_P$ and $\alpha_6 = 0$ in Theorem 2, we obtain Theorem 3 of Harinath[3].

(3) On taking $T = I_P$ and $\alpha_6 = \alpha_7 = 0$ in Theorem 1, we get Theorem 5 of Harinath[3].

(4) On making $T = I_M$ (identity map of M) and $\alpha_2 = \alpha_5$, $\alpha_3 = \alpha_4$, and $\alpha_6 = \alpha_7 = 0$ in Theorem 3, we get Theorem 4 of Fisher[2].

(5) If we take $T = I_M$ and choose $\alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = \alpha_7 = 0$ and $\alpha_3 = \alpha_4 = \frac{1}{2}$ in Theorem 3, we get Theorem 3 of Fisher[2].

(6) In case $T = I_M$, $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_6 = \alpha_7 = 0$ and $\alpha_2 = \alpha_5 = \frac{1}{2}$ in Theorem 3, we get Theorem 2 of Fisher[2].

(7) On taking $T = I_M$, $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 0$ in Theorem 3, we obtain Theorem 1 of Fisher[2].

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