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COMMON FIXED POINTS OF TWO PAIRS OF COMMUTING MAPPINGS

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A common fixed point theorem for two pairs of commuting mappings satisfying Meir and Keeler type condition is obtained. Our work generalizes results due to Fisher, Meir and Keeler, Park and Bae.

1. INTRODUCTION

Meir and Keeler[2] obtained a remarkable generalization of the Banach contraction principle. In this paper we generalize Meir and Keeler theorem to obtain a common fixed point theorem for two pairs of commuting mappings satisfying Meir and Keeler type condition. Theorem 2 due to Fisher[1] and many other important results can be obtained as a special case of our theorem.

2. RESULTS

Theorem—Let P and S be commuting mappings and Q and T be commuting mappings of a complete metric space (X, d) into itself satisfying the conditions:

Given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, $\delta(\epsilon)$ being a nondecreasing function of ϵ, such that for all x, y in X

$$\epsilon \leqslant \max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} < \epsilon + \delta$$

$$\Rightarrow d(Px, Qy) < \epsilon \qquad \ldots(1)$$

$$Px = Qy \text{ whenever } Px = Sx, Qy = Ty. \qquad \ldots(2)$$

If the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a unique common fixed point z. Further, z is the unique common fixed point of P and S and of Q and T.

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PROOF : First, with the help of (1), we note that for all x, y in X such that $Px \neq Sx$, $Qy \neq Ty$

$$d(Px, Qy) < \max \{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\}. \quad \ldots(3)$$

Secondly, the nondecreasing character of $\delta(\epsilon)$ implies that given $\epsilon > 0$, there exists $\epsilon' > 0$, such that $\epsilon' < \epsilon < \epsilon' + \delta(\epsilon')$ or equivalently

$$\max \{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} = \epsilon$$

$$\Rightarrow d(Px, Qy) < \epsilon', \epsilon' < \epsilon. \quad \ldots(4)$$

Let x_0 be an arbitrary point in X. Choose a point x_1 and then a point x_2 in X such that $Px_0 = Tx_1$ and $Qx_1 = Sx_2$. This can be done since the range of T contains the range of P and the range of S contains the range of Q. In general having chosen the point x_{2n} choose a point x_{2n+1} and then a point x_{2n+2} such that $Px_{2n} = Tx_{2n+1}$ and $Qx_{2n+1} = Sx_{2n+2}$. Then

$$d(Px_{2n}, Qx_{2n+1}) < \max \{d(Sx_{2n}, Tx_{2n+1}), d(Px_{2n}, Sx_{2n}), d(Qx_{2n+1}, Tx_{2n+1})\}$$

$$= \max \{d(Qx_{2n-1}, Px_{2n}), d(Qx_{2n+1}, Px_{2n})\}$$

$$= d(Qx_{2n-1}, Px_{2n}).$$

Similarly

$$d(Qx_{2n-1}, Px_{2n}) < d(Px_{2n-2}, Qx_{2n-1}).$$

From the last two inequalities we conclude that both $d(Px_{2n}, Qx_{2n+1})$ and $d(Qx_{2n+1}, Px_{2n+2})$ are monotone decreasing sequences of positive real numbers. We claim that $\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = 0 = \lim_{n\to\infty} d(Qx_{2n+1}, Px_{2n+2})$. For, if not, suppose for instance $\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = r$, $r > 0$. Then given $\delta > 0$, there exists a positive integer N such that for each integer $m \geq N$, we have

$$r \leq d(Px_{2m}, Qx_{2m+1}) < r + \delta \quad \ldots(5)$$

or

$$r \leq \max \{d(Sx_{2m+2}, Tx_{2m+1}), d(Px_{2m+2}, Sx_{2m+2}),$$

$$d(Qx_{2m+1}, Tx_{2m+1})\} < r + \delta. \quad \ldots(6)$$

Selecting δ in (6) in accordance with (1), for each $m \geq N$ we obtain $d(Px_{2m+2}, Qx_{2m+1}) < r$, and so $d(Px_{2m+2}, Qx_{2m+3}) < r$, which contradicts (5). Therefore $\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = 0$ and similarly $\lim_{n\to\infty} d(Qx_{2n+1}, Px_{2n+2}) = 0$. From these two limits and condition (1) it follows easily that

$$\{Px_0, Qx_1, Px_2, Qx_3, \ldots, Px_{2n}, Qx_{2n+1}, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X. Hence the sequences

$$\{Px_{2n}\} = \{Tx_{2n+1}\} \text{ and } \{Qx_{2n+1}\} = \{Sx_{2n+2}\}$$

coverge to the point z.

Let us now suppose that the mapping S is continuous. Then since the mappings P and S commute, the sequences $\{SSx_{2n}\}$ and $\{PSx_{2n}\}$ converge to the point Sz. We claim that $Sz = z$. For if $z \neq Sz$, then the inequality

$$d(PSx_{2n}, Qx_{2n+1}) < \max\{d(SSx_{2n}, Tx_{2n+1}), d(PSx_{2n}, SSx_{2n}), d(Qx_{2n+1}, Tx_{2n+1})\}$$

on letting $n \to \infty$ and in view of (4) leads to $d(Sz, z) < d(Sz, z)$, a contradiction. Therefore $Sz = z$. Similarly the inequality

$$d(Pz, Qx_{2n+1}) < \max\{d(Sz, Tx_{2n+1}), d(Pz, Sz), d(Qx_{2n+1}, Tx_{2n+1})\}$$

on letting $n \to \infty$ yields $z = Pz$. This means that there exists a point z' in X such that $Sz = z = Pz = Tz'$, since the range of T contains the range of P. And so

$$QTz' = Qz = TQz' \qquad \ldots(7)$$

since Q and T commute. Further $z = Qz'$, otherwise

$$\begin{aligned} d(z, Qz') &= d(Pz, Qz') \\ &< \max\{d(Sz, Tz'), d(Pz, Sz), d(Qz', Tz')\} \\ &= d(z, Qz'), \end{aligned}$$

a contradiction. Therefore $z = Qz' = Tz'$. Equations (7) then lead to $Qz = Tz$, which then leads to $z = Qz = Tz$, for otherwise

$$\begin{aligned} d(z, Qz) &= d(Pz, Qz) \\ &< \max\{d(Sz, Tz), d(Pz, Sz), d(Qz, Tz)\} \\ &= d(z, Qz) \end{aligned}$$

a contradiction.

We have therefore proved that z is a common fixed point of P, Q, S and T.

Similarly it can be proved that z is a common fixed point of P, Q, S and T if T is continuous instead of S.

Next suppose that the mapping P is continuous. Then since P and S commute, the sequences $\{PPx_{2n}\}$ and $\{SPx_{2n}\}$ converge to Pz. Now the ineqality

$$d(PPx_{2n}, Qx_{2n+1}) < \max \{d(SPx_{2n}, Tx_{2n+1}), d(PPx_{2n}, SPx_{2n}), d(Qx_{2n+1}, Tx_{2n+1})\}$$

on letting $n \to \infty$ and in view of (4) leads to $z = Pz$.

It follows from $z = Pz$ that there exists a point z' in X such that $z = Pz = Tz'$ and eqns. (7) hold, since the range of T contains the range of P. Then from

$$d(PPx_{2n}, Qz') < \max \{d(SPx_{2n}, Tz'), d(PPx_{2n}, SPx_{2n}), d(Qz', Tz')\}$$

on letting $n \to \infty$ we obtain $z = Qz'$. And so

$$Tz = TQz' = QTz' = Qz.$$

Now the inequality

$$d(Px_{2n}, Qz) < \max \{d(Sx_{2n}, Tz), d(Sx_{2n}, Px_{2n}), d(Qz, Tz)\}$$

on letting $n \to \infty$ and in view of (4) yields $z = Qz = Tz$. And this means that there is a point z'' in X such that $z = Qz = Sz''$, since the range of S contains the range of Q. Then

$$d(Pz'', z) = d(Pz'', Qz) < \max \{d(Sz'', Tz), d(Pz'', Sz''), d(Qz, Tz)\}$$

yields $z = Pz'' = Sz''$. The commutativity of P and S now leads to $PSz'' = Sz = SPz'' = Pz = z$.

We have therefore proved that z is again a common fixed point of P, Q, S and T.

Similarly we can prove that z is again a common fixed point of P, Q, S and T if Q is continuous instead of P.

Finally, if possible, suppose that w is a second common fixed point of P and S. Then

$$d(w, z) = d(Pw, Qz) < \max \{d(Sw, Tz), d(Pw, Sw), d(Qz, Tz)\} = d(w, z)$$

a contradiction. Hence z is the uniqe common fixed point of P and S. Similarly z is the unique common fixed point of Q and T. This completes the proof of the theorem.

Corollary—Let P and S be commuting mappings and Q and T be commuting mappings of a complete metric space (X, d) into itself satisfying the conditions:

Given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, $\delta(\epsilon)$ being a nondecreasing function of ϵ, such that for all x, y in X

$$\epsilon \leqslant \max \{d(Sx, Ty), d(Sx, Px), d(Qy, Ty),$$

$$\tfrac{1}{2} d(Px, Ty), \tfrac{1}{2} d(Sx, Qy)\} < \epsilon + \delta$$

$$\Rightarrow d(Px, Qy) < \epsilon \qquad \ldots(8)$$

$$Px = Qy \text{ whenever } Px = Sx, Qy = Ty. \qquad \ldots(9)$$

If the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a unique common fixed point z. Further, z is the unique common fixed point of P and S and of Q and T.

PROOF : For all x, y in X we have

$$\tfrac{1}{2} d(Sx, Qy) \leqslant \tfrac{1}{2} [d(Sx, Px) + d(Px, Qy)]$$

and

$$\tfrac{1}{2} d(Px, Ty) \leqslant \tfrac{1}{2} [d(Px, Qy) + d(Qy, Ty)].$$

Therefore condition (8) is equivalent to

$$\epsilon \leqslant \max \{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} < \epsilon + \delta$$

$$\Rightarrow d(Px, Qy) < \epsilon.$$

The conditions of the main theorem are therefore satisfied and the result of the corollary follows.

Remark 1 : In the above theorem set $\delta(\epsilon) = \dfrac{2(1-c)\epsilon}{c}$, $0 < c < 1$. Then $\delta(\epsilon)$ is nondecreasing function of ϵ and given $\epsilon > 0$, we have

$$\epsilon' < \epsilon = \epsilon' + \tfrac{1}{2} \delta(\epsilon') < \epsilon' + \delta(\epsilon')$$

by taking $\epsilon' = c\epsilon$. The condition (1) of the theorem then reduces to

$$d(Px, Qy) \leqslant c \max \{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\}$$

and we obtain Theorem 2 of Fisher[1] as a special case of our theorem.

Remark 2 : In the above theorem if we take $P = Q$, $S = T =$ identity mapping and max $\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} = d(Sx, Ty)$, then the assumption on nondecreasing character of $\delta(\epsilon)$ can be dropped and we obtain the Theorem of Meir and Keeler[2] as a special case of our theorem.

Remark 3 : In the above theorem if we take $P = Q$, $S = T$, T to be a continuous mapping and max $\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} = d(Sx, Ty)$, then the assumption on nondecreasing character of $\delta(\epsilon)$ can be dropped and we obtain Theorem 2.4 of Park and Bae[3] as a special case of our theorem.

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