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ISHIKAWA ITERATES FOR A PAIR OF MAPS

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The concept of Ishikawa iterates associated with a pair of self-maps on a subset of a normed (metric) linear space is studied and certain fixed point theorems are obtained.

In this paper we extend the notion of Ishikawa iterates to a pair of self-maps on a subset of a normed linear space and show that under certain contractive conditions on the maps the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps and thereby obtain certain generalizations of some of the results of Naimpally and Singh[1].

Throughout this paper, unless otherwise stated X is a normed linear space; C is a nonempty convex subset of X; T_1, T_2 are self-maps on C; $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers in [0, 1]; R^+ is the set of all non-negative real numbers; and ϕ $R^+ \to R^+$ is an increasing map with $\phi(t+) < t \; \forall \; t \in (0, \infty)$ and $\lim_{t \to +\infty} (t - \phi(t)) > 0$.

The sequence $\{x_n\}$ of Ishikawa iterates associated with the pair (T_1, T_2) is defined as follows :

$$x_0 \in C,$$

$$x_{n+1} = (1 - \alpha_n)\; x_n + \alpha_n \; T_2 y_n \; (n > 0) \qquad \ldots(1)$$

where

$$y_n = \beta_n \; T_1 x_n + (1 - \beta_n)\; x_n \;\; (n \geqslant 0). \qquad \ldots(2)$$

For the sake of brevity we denote $\|T_1 x_n - T_2 y_n\|$ by s_n and $\|x_n - T_2 y_n\|$ by t_n.

Theorem 1—Suppose that

$$\|T_1 x - T_2 y\| \leqslant \phi(\max\{\|x - y\|, \|x - T_1 x\|, \|y - T_2 y\|, \|T_1 x - y\| + \|x - T_2 y\|\}) \qquad \ldots(3)$$

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for all x, y in C. Suppose also that the sequence $\{\alpha_n\}$ is bounned away from zero and the sequence $\{x_n\}$ converges to some $p \in C$. Then the sequences $\{y_n\}$, $\{T_1 x_n\}$ and $\{T_2 y_n\}$ all converge to p and p is a common fixed point of T_1 and T_2.

PROOF : From eqn. (1) we have

$$x_{n+1} - x_n = \alpha_n (T_2 y_n - x_n) (n \geqslant 0).$$

Since $\{x_n\}$ is convergent and $\{\alpha_n\}$ is bounded away from zero it is clear that $\{\|T_2 y_n - x_n\|\}$ converges to zero. Since $\{x_n\}$ converges to p, the sequence $\{T_2 y_n\}$ must converges to p. From eqn. (2) we have $x_n - y_n = \beta_n (x_n - T_1 x_n)$ so that

$$\|x_n - y_n\| \leqslant \|x_n - T_1 x_n\| \leqslant s_n + t_n.$$

Again from equation (2) we have $y_n - T_1 x_n = (1 - \beta_n)(x_n - T_1 x_n)$

so that

$$\|y_n - T_1 x_n\| \leqslant \|x_n - T_1 x_n\| \leqslant s_n + t_n.$$

Once again from eqn. (2) we have

$y_n - T_2 y_n = \beta_n (T_1 x_n - T_2 y_n) + (1 - \beta_n)(x_n - T_2 y_n)$ so that $\|y_n - T_2 y_n\| \leqslant \| \beta_n (T_1 x_n - T_2 y_n)\| + \| (1 - \beta_n)(x_n - T_2 y_n)\| \leqslant s_n + t_n$.

Now taking $x = x_n$ and $y = y_n$ in inequality (3) and making use of the above three inequalities we obtain

$$s_n \leqslant \phi (s_n + 2t_n) \ (n \geqslant 0) \qquad \ldots(4)$$

i. e.

$$(s_n + 2t_n) - \phi (s_n + 2t_n) \leqslant 2t_n \ (n \geqslant 0).$$

Since $\{t_n\}$ converges to zero and $\lim_{t \to +\infty} (t - \phi(t)) > 0$, it follows from the above inequality that the sequence $\{s_n\}$ is bounded. If $\{s_n\}$ has a convergent subsequence with a positive limit l then inequality (4) would imply that $l \leqslant \phi (1 +)$ which would be a contradication since $\phi (t +) < t \ \forall \ t \in (0, \infty)$. Hence $\{s_n\}$ must converge to zero. Since $\{T_2 y_n\}$ converges to p it now follows that $\{T_1 x_n\}$ also converges to p. From eqn. (2) it is now evident that $\{y_n\}$ converges to p. Taking $x = p$ and $y = y_n$ in inequality (3) and then taking limits on both sides as $n \to \infty$ we obtain $\|T_1 p - p\| \leqslant \phi (\|T_1 p - p\| +)$ so that $T_1 p = p$.

Taking $x = y = p$ in inequality (3) and making use of the fact that $T_1 p = p$ we obtain

$$\|p - T_2 p\| \leqslant \phi (\|p - T_2 p\|)$$

so that $T_2 p = p$.

Remark 1 : Theorem 1.2 of Naimpally and Singh[1] is a special case of Theorem 1 with $T_1 = T_2$ and $\phi(t) = \alpha t$, α being a constant in [0, 1).

Remark 2 : It was observed by Sastry and Naidu[2] that in a metric linear space (X, d) with ball convexity $(B.C.)$ the function $d(tx, 0)$ is an increasing function of t on R^+ for any $x \in X$. By making use of this result one can easily extend Theorem 1 to metric linear spaces with $B.C.$ as follows :

Theorem 2— Let (X, d) be a metric linear space with $B.C.$, C be a nonempty convex subset of X; T_1, T_2 be self-maps on C and $\{x_n\}$, $\{y_n\}$ be sequences in C defined by eqns. (1) and (2) respectively. Suppose that

$$d(T_1 x, T_2 y) \leqslant \phi \max \{d(x, y), d(x, T_1 x), d(y, T_2 y), d(T_1 x, y) + d(x, T_2 y)\}$$

for all x, y in C. Suppose also that the sequence $\{\alpha_n\}$ is bounded away from zero and the sequence $\{x_n\}$ converges to some $p \in C$. Then the sequences $\{y_n\}$, $\{T_1 x_n\}$ and $\{T_2 y_n\}$ all converge to p and $T_1 p = p = T_2 p$.

Theorem 3—Suppose that

$$\|T_1 x - T_2 y\| \leqslant \|x - y\| + \phi(\|T_1 x - y\| + \|x - T_2 y\|) \qquad ...(5)$$

for all x, y in C. Further suppose that the sequence $\{x_n\}$ converges to some $p \in C$, the sequence $\{\alpha_n\}$ is bounded away from zero and $\overline{\lim_{t \to \infty}} \beta_n < 1$. Then the sequences $\{y_n\}$, $\{T_1 x_n\}$ and $\{T_2 y_n\}$ all converge of p and $T_1 p = p = T_2 p$.

PROOF : Proceeding almost along the same lines as in the proof of Theorem 1 it can be shown that the sequence $\{t_n\}$ converges to zero, the sequence $\{T_2 y_n\}$ converges to p

$$\|x_n - y_n\| \leqslant \beta_n (s_n + t_n) \qquad ...(6)$$

and

$$\|T_1 x_n - y_n\| \leqslant (1 - \beta_n)(s_n + t_n). \qquad ...(7)$$

Taking $x = x_n$ and $y = y_n$ on both sides of inequality (5) and makinin use of inequalities (6) and (7) we obtain

$$s_n \leqslant \beta_n (s_n + t_n) + \phi((1 - \beta_n)(s_n + t_n) + t_n) \ (n \geqslant 0).$$

i. e.

$$l_n - \phi(l_n) \leqslant 2t_n \ (n \geqslant 0)$$

where $l_n = (1 - \beta_n)(s_n + t_n) + t_n$.

Since $\{t_n\}$ converges to zero, $\lim_{t \to +\infty} [t - \phi(t)] > 0$ and $\phi(t+) < t \; \forall \; t \in (0, \infty)$ it can be shown as in the proof of Theorem 1 that $\{l_n\}$ converges to zero. Hence $\{(1 - \beta_n) s_n\}$ converges to zero. Since $(\overline{\lim_{t \to \infty}} \beta_n) < 1$, the sequence $\{(1 - \beta_n)\}$ is bounded away from zero and hence the sequence $\{s_n\}$ must converge to zero.

The rest of the proof follows as in Theorem 1.

Remark 3 : In Theorem 3 the conclusion that p is a common fixed point of T_1 and T_2 remins unaffected if the hypothesis on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ is relaxed by assuming one of the following two conditions :

(i) $\overline{\lim_{n \to \infty}} \alpha_n > 0$ and $\overline{\lim_{n \to \infty}} \beta_n < 1$.

(ii) $\underline{\lim_{n \to \infty}} \alpha_n > 0$ and $\underline{\lim_{n \to \infty}} \beta_n < 1$.

Lemma —Let Y be a nonempty subset of X and T be a self-map on Y satisfying one of the following four conditions :

(i) $\|x - Tx\| + \|y - Ty\| \leqslant \alpha \|x - y\|$ for all x, y in Y and for some consatnnt α in [0, 2].

(ii) $\|x - Tx\| + \|y - Ty\| \leqslant \alpha (\|x - y\| + \|Tx - y\| + \|x - Ty\|)$

for all x, y in Y and for some constant α in [0, 1).

(iii) $\|Tx - Ty\| + \|x - Tx\| + \|y - Ty\| \leqslant \alpha (\|Tx - y\| + \|x - Ty\|)$ for all x, y in Y and for some constant α in [0, 2).

(iv) $\|Tx - Ty\| \leqslant \alpha \max \{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|Tx - y\| + \|x - Ty\|\}$

for all x, y in Y and for come constant α in [0, 1).

Then there exists a constant β in [0, 1) such that

$$\|Tx - Ty\| \leqslant \|x - y\| + \beta (\|Tx - y\| + \|x - Ty\|)$$

for all x, y in Y.

PROOF : We have

$$2\|Tx - Ty\| = \|Tx - Ty\| + \|Tx - Ty\|$$

$$\leqslant (\|Tx - x\| + \|x - Ty\|) + (\|Tx - y\| + \|y - Ty\|)$$

so that

$$\|Tx - Ty\| \leqslant \tfrac{1}{2}\,[\,(\|x - Tx\| + \|y - Ty\|)$$

$$+ (\|Tx - y\| + \|x - Ty\|)] \qquad \ldots(8)$$

for all x, y in Y.

From inequality (8) it is clear that either

$$\|Tx - Ty\| \leqslant \|x - y\| + \tfrac{1}{2}\,(\|Tx - y\| + \|x - Ty\|)\ \forall\ x, y \in Y$$

or

$$\|Tx - Ty\| \leqslant \|x - y\| + \frac{(1 + \alpha)}{2}\,(\|Tx - y\| + \|x - Ty\|)\ \forall\ x, y \in Y$$

according as (i) or (ii) holds.

Again we have

$$2\|Tx - Ty\| = \|Tx - Ty\| + \|Tx - Ty\|$$

$$\leqslant \|Tx - Ty\| + (\|Tx - x\| + \|x - y\| + \|y - Ty\|)$$

so that

$$\|Tx - Ty\| \leqslant \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\,(\|Tx - Ty\| + \|x - Tx\| + \|y - Ty\|)$$

for all x, y in Y.

Suppose (iii) holds. Then from the above inequality we have

$$\|Tx - Ty\| \leqslant \|x - y\| + \frac{\alpha}{2}\,(\|Tx - y\| + \|x - Ty\|)\ \forall\ \ x, y \in Y.$$

Since

$$\|x - Tx\| \leqslant \|x - y\| + \|y - Tx\|$$

and

$$\|y - Ty\| \leqslant \|x - y\| + \|x - Ty\|$$

it is clear that $\|Tx - Ty\| \leqslant \|x - y\| + \alpha\,(\|x - Ty\| + \|Tx - y\|)$

for all x, y in Y when (iv) holds.

Hence the lemma.

Remark 4 : In view of the Lemma it is clear that Theorem 1.4 of Naimpally and Singh[1] is a special case of Theorem 3.

Theorem 4—Suppose that

$$\|T_1 x - T_2 y\| \leqslant \|x - y\| + \max \{ \phi (\|T_1 x - y\| + \|x - T_2 y\|),$$

$$\Psi (\|x - T_1 x\| + \|y - T_2 y\|)\} \quad .. (9)$$

for all x, y in C, where $\Psi : R^+ \to R^+$ is an increasing function with $\Psi(t+) < t \; \forall \; t \in (0, \infty)$ and $\underline{\lim}_{n \to \infty} (a_n - \Psi(\gamma_n a_n)) > 0$, whenever $\{a_n\}$ is a sequence in R^+ diverging to $+\infty$ and $\{\gamma_n\}$ is a sequence in $[1, \infty)$ converging to 1. Further suppose that $\{x_n\}$ converges to some $p \in C$, $\{\alpha_n\}$ is bounded away from zero and $\{\beta_n\}$ converges to zero. Then the sequences $\{y_n\}$, $\{T_1 x_n\}$ and $\{T_2 y_n\}$ all coverge to p and $T_1 p = p = T_2 p$.

PROOF : As in the proof of Theorem 1 it can be seen that $\{t_n\}$ converges to zero and $\{T_2 y_n\}$ converges to p.

Clearly we have

$$\|x_n - T_1 x_n\| \leqslant s_n + t_n. \quad ...(10)$$

From eqns. (1) and (2) it is clear that inequalities (6) and (7) hold and further

$$\|y_n - T_2 y_n\| \leqslant \beta_n s_n + (1 - \beta_n) t_n. \quad ...(11)$$

Taking $x = x_n$ and $y = y_n$ in inequality (9) and then making use of inequalities (6), (7), (9) and (11) we get

$$s_n \leqslant \beta_n (s_n + t_n) + \max \{ \phi ((1 - \beta_n) s_n + (2 - \beta_n) t_n),$$

$$\Psi ((1 + \beta_n) s_n + (2 - \beta_n) t_n)\} \; (n \geqslant 0).$$

i. e.

$$(1 - \beta_n) s_n - \beta_n t_n \leqslant \max \{\phi ((1 - \beta_n) s_n + (2 - \beta_n) t_n),$$

$$\Psi ((1 + \beta_n) s_n + (2 - \beta_n) t_n)\} \; (n \geqslant 0). \quad ...(12)$$

Hence for each $n \geqslant 0$ we have either

$$g_n - \phi (g_n) \leqslant 2 t_n \quad ...(13)$$

or

$$h_n - \Psi(\gamma_n h_n \leqslant \left(\frac{(2-\beta_n)(1-\beta_n)}{(1+\beta_n)} + \beta_n \right) t_n \qquad \ldots(14)$$

where

$$g_n = (1-\beta_n)\, s_n + (2-\beta_n)\, t_n$$

$$h_n = (1-\beta_n)\, s_n + \frac{(2-\beta_n)(1-\beta_n)}{(1+\beta_n)}\, t_n$$

and

$$\gamma_n = \frac{(1+\beta_n)}{(1-\beta_n)} .$$

Suppose the sequence $\{s_n\}$ is unbounded.

Then $\{s_n\}$, being a sequence of non-negative real numbers, admits a subsequence $\{s_{n_k}\}$ which diverges to $+\infty$.

If inequality (13) with n_k in the place of n holds for infinitely many k then by taking limits on both sides of this inequality as $k \to \infty$ over these values one can conclude (in view of the fact that the sequences $\{\beta_n\}$ and $\{t_n\}$ converge to zero) that $\overline{\lim_{t \to +\infty}}$ $(t - \phi(t)) \leqslant 0$ which is a contrrdication.

If inequality (14) with n_k in the place of n holds for infinitely many k then one can conclude that $\overline{\lim_{k \to \infty}}$ $(h_{n_k} - \Psi(\gamma_{n_k} h_{n_k})) \leqslant 0$ which is a contradiction to the hypothesis on Ψ since $h_{n_k} \to +\infty$ as $k \to +\infty$ and $\gamma_{n_k} \to 1+$ as $k \to +\infty$.

Hence the sequence $\{s_n\}$ must be bounded.

If $\{s_n\}$ has a convergent subsequence with a positive limit l then from inequality $(X/2)$ it is easy to show that

$$l \leqslant \max\{\phi(l+), \Psi(l+)\}$$

which is a contradiction since $\phi(t+) < t$ and $\Psi(t+) < t \; \forall \; t \in (0, \infty)$.

Hence the sequence $\{s_n\}$ must converge to zero.

The rest of the proof follows as in Theorem 1.

Remark 5 : In Theorems 1, 2 and 4 the conclusion that p is a common fixed point of T_1 and T_2 remains unaffected if the hypothesis on $\{\alpha_n\}$ is relaxed by assuming that $\overline{\lim_{t \to \infty}}$ $\alpha_n > 0$.

Recently it came to the notice of the authors that Das and Debata[3] extended Ishikawa iteration scheme to a finite family of self-maps on a subset of a Hilbert spaces, which of course has a natural extension to normed linear spaces.

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