Indian J. pure appl. Math., **17** (1) : 21-25, January 1986

AN EQUIVALENCE THEOREM CONCERNING SYMMETRY RECURSIVITY AND CYCLIC SYMMETRY

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(*Received* 16 *June* 1984; *after revision* 6 *August* 1985)

An equivalence theorem concerning the three properties of $E_n : S_n \to R =]-\infty, \infty[$, namely, symmetry, recursivity and cyclic symmetry, is proved.

1. INTRODUCTION

Let for $n = 2, 3, 4, \ldots$

$$\Gamma_n = \{ (p_1, p_2, \ldots, p_n) : \; p_i \geqslant 0, \quad i = 1, 2, \ldots, n; \; \sum_{i=1}^{n} p_i = 1 \}$$

denote the set of all finite n-component complete probability distributions with non-negative elements and S_n, $(n = 2, 3, \ldots)$, the set of all $2n$-tuples of the form $(p_1, p_2, \ldots, p_n; q_1, q_2 \ldots, q_n)$ with $(p_1, p_2, \ldots, p_n) \in \Gamma_n$, $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ such that $p_i = 0$ for all those indices i for which $q_i = 0$, $1 \leqslant i \leqslant n$. Let $E_n : S_n \to R =]-\infty, \infty[$, $n = 2, 3, \ldots$ be a sequence of functions. We state below only those properties of E_n on which the whole discussion is based.

I_n (Recursivity) $E_n : S_n \to R$ is recursive, that is, for all $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$ with $p_1 + p_2 > 0$,

$$E_n (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = E_{n-1} (p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) + (p_1 + p_2) \; E_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right), \quad p_1 + p_2 > 0. \qquad \ldots(1)$$

Notice that, in S_n, $p_1 + p_2 > 0 \Rightarrow q_1 + q_2 > 0$.

II_n (Symmetry) $E_n : S_n \to R$ is symmetirc, that is, for all

$$(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n,$$

$$E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = E_n(p_{\pi(1)}, p_{\pi(2)}, \cdots p_{\pi(n)}; q_{\pi(1)}, q_{\pi(2)}, \cdots q_{\pi(n)}) \quad \ldots(2)$$

where $\pi(1), \pi(2), \ldots, \pi(n)$ is an arbitrary permutation of 1, 2, ..., n.

II_n^* (Cyclic Symmetry) $E_n : S_n \to R$ is cyclic, that is, for all

$$(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$$

$$E_n(p_1, p_2, \ldots, p_{n-1}, p_n; q_1, q_2, \ldots, q_{n-1}, q_n) = E_n(p_n, p_1, p_2, \ldots, p_{n-1}; q_n, q_1, q_2, \ldots, q_{n-1}). \quad \ldots(3)$$

Obviously II_2 and II_2^* are 'equivalent' and II_n^* $(n \geqslant 3)$ is weaker than II_n $(n \geqslant 3)$ in the 'strict sence'.

The object of this paper is to prove an equivalence theorem concerning the above mentioned three propeties.

2. An Equivalence Theorem

The main result of this paper is the following :

Theorem—The following two systems are equivalent :

System 1 : I_n $(n = 3, 4, \ldots)$ and II_n $(n = 2, 3, \ldots)$

System 2 : I_n $(n = 3, 4, \ldots)$ and II_n^* $(n = 3, 4, \ldots)$.

We first prove some lemmas.

Lemma $1 - I_3$ *and* II_3^* *imply*

$$E_2(1, 0; 1, 0) = E_2(0, 1; 0, 1) = 0 \quad \ldots(4)$$

$$E_2(p_1, p_2; q_1, q_2) = E_2(p_2, p_1; q_2, q_1). \quad \ldots(5)$$

Proof : Equation (4) is a simple consequence of I_3 and II_3^*. Hence its proof is omitted.

Let $(p_1, p_2; q_1, q_2) \in S_2$. If $q_1 = 0$, then $p_1 = 0$ and $p_2 = q_2 = 1$; if $q_1 = 1$, then $q_2 = 0$ and consequently $p_2 = 0$, $p_1 = 1$. In both these situations, (5) follows from (4). If $0 < q_1 < 1$, then $0 < q_2 < 1$. In this case

$$E_2(p_1, p_2; q_1, q_2) \overset{(4)}{=} \begin{cases} E_2(1, 0; q_1, q_2) + E_2(1, 0; 1, 0) & \text{if } p_2 = 0 \\ E_2(1, 0; 1, 0) + E_2(p_1, p_2; q_1, q_2) & \text{if } 0 < p_2 \leqslant 1 \end{cases}$$

$$\overset{(1)}{=} \begin{cases} E_3(0, 1, 0; 0; q_1, 0, q_2) & \text{if } p_2 = 0 \\ E_3(p_1, p_2, 0; q_1, q_2, 0) & \text{if } 0 < p_2 \leqslant 1 \end{cases}$$

$$= \begin{cases} E_3(0, 1, 0; q_2, q_1, 0) & \text{if } p_2 = 0 \text{ (using (3) once)} \\ E_3(p_2, 0, p_1; q_2, 0, q_1) & \text{if } 0 < p_2 \leqslant 1 \\ & \text{(using (3) twice)} \end{cases}$$

$$\overset{(1)}{=} \begin{cases} E_2(1, 0; 1, 0) + E_2(0, 1; q_2, q_1) & \text{if } p_2 = 0 \\ E_2(p_2, p_1; q_2, q_1) + p_2 E_2(1, 0; 1, 0) & \text{if } 0 < p_2 \leqslant 1 \end{cases}$$

$$\overset{(4)}{=} E_2(p_2, p_1, q_2, q_1).$$

Lemma 2—I_n $(n = 3, 4, \ldots)$ and II_n^* $(n = 3, 4, \ldots)$ imply

$$E_n(a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n; b_1, b_2, \ldots, b_{k-1}, 0, b_{k+1}, \ldots, b_n)$$
$$= E_n(a_1, a_2, \ldots, 0, a_{k-1}, a_{k+1}, \ldots, a_n; b_1, b_2, \ldots, 0, b_{k-1}, b_{k+1}, \ldots, b_n) \quad \ldots(6)$$
$$a_1 > 0; k = 2, 3, \ldots, n-1; n = 3, 4, \ldots$$

$$E_n(a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n) = E_n(a_2, a_1, a_3, \ldots, a_n; b_2, b_1, b_3, \ldots, b_n) \quad \ldots(7)$$
$$0 < a_1 + a_2 \leqslant 1; n = 3, 4, \ldots$$

$$E_n(a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n) = E_n(a_1, a_3, a_2, \ldots, a_n; b_1, b_3, b_2, \ldots, b_n) \quad \ldots(8)$$
$$a_1 > 0; n = 3, 4, \ldots$$

$$E_n(a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n)$$
$$= E_n(a_1, a_{n-j+1}, a_{n-j+2}, \ldots, a_n, a_2, a_3, \ldots, a_{n-j}; b_1, b_{n-j+1}, b_{n-j+2}, \ldots, b_n, b_2, b_3, \ldots, b_{n-j}) \quad \ldots(9)$$
$$a_1 > 0; j = 1, 2, \ldots, n-2; n = 3, 4, \ldots$$

PROOF : Equation (6) follows by repeated use of I_n and (4); (7) from I_n and (5); (8) can be proved by following the procedure explained in the proof of Proposition 2.3.5, pages 59-60 in Aczél and Daróczy[1]; and (9) by using (3) and (7) alternatively depending upon the value of j. For instance, if $j = 1$, use (3) and (7) in this order only once.

Lemma 3 – I_n $(n = 3, 4, ...)$ and II_n^* $(n = 3, 4, ..)$ imply

$$E_n(p_1, p_2, p_3, .., p_n; q_1, q_2, q_3, .., q_n)$$

$$= E_n(p_2, p_1, p_3, ..., p_n; q_2, q_1, q_3, ..., q_n) \qquad ...(10)$$

$$n = 3, 4, ...$$

PROOF : Let $(p_1, p_2, ..., p_n; q_1, q_2, ..., q_n) \in S_n$ where n is any arbitrarily fixed integer $\geqslant 3$. If $0 < p_1 + p_2 \leqslant 1$, then (10) turns out to be (7). If $p_1 + p_2 = 0$, then $p_1 = p_2 = 0$. Let p_d, $3 \leqslant d \leqslant n$, be the first positive element among p_i's in $(p_1, p_2, ..., p_n; q_1, q_2, ..., q_n) \in S_n$. Then $q_d > 0$ and have

$$E_n(p_1, p_2, p_3, ..., p_n; q_1, q_2, q_3, ..., q_n) = E_n(0, 0, ..., p_d, ..., p_n; q_1, q_2, .., q_d, ... q_n) \qquad ...(11)$$

If $q_1 = 0$ or $q_2 = 0$ or $q_1 = q_2 = 0$, then apply in succession to the right hand side of (11); (3) $n - d + 1$ times; then (6) to interchange q_1 and q_2; and finally (3) again but now only d times. Then (10) follows.

If $0 < q_1 < 1$, $0 < q_2 < 1$, then apply in succession to the right hand side of (11); (3) $n - d + 2$ times; (7); (9) with $j = d - 1$; (8); (9) with $j = n - d$; and finally (3) d times. This proves (10).

Proof of the main theorem—System 1 obviously implies System 2. The fact that System 2 implies System 1 follows from I_n $(n = 3, 4, ...,)$, (5), (10), and II_n^* $(n = 3, 4, ...)$ because (10) and II_n^* $(n = 3, 4, ...)$ imply II_n $(n = 2, 3, ...)$.

The proofs of Lemmas 2 and 3 are written descriptively to avoid writing lengthy mathemetical calculations.

Remarks (a) – In the theory of permutation groups, it is known that the full permutation group $\mathscr{S}_n$ can be generated by a cycle of length n and a transport. Equation (3) gives rise to

$$E_n(p_1, p_2, ..., p_n; q_1, q_2, ..., q_n)$$

$$= E_n(p_2, p_3, ..., p_n, p_1; q_2, q_3, ..., q_n, q_1)$$

which may be identified with the cycle [1 2 3 ... n] of length n. Equation (10) may be identified with the transport [12], a cycle of length 2, indicating the interchange of p_1 with p_2 and of q_1 with q_2. This is the reason behind proving (10).

(b) Nath and Kaur[2] have already discussed the cyclic symmetry in connection with the Shannon entropy[3]. In many axiomatic characterizations of symmetric information-theoretic measures like inaccuracy, directed divergence etc; properties such as continuity, symmetry and recursivity have been assumed. The equivalence theorem, proved above, says that in all such axiomatic characterizations, it is enough to assume recursivity and cyclic symmetry in place of recursivity and symmetry.

ACKNOWLEDGEMENTS

The authors are indebted to the referees for their constructive comments.

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