AN EQUIVALENCE THEOREM CONCERNING SYMmetry
RECURSIVITY AND CYCLIC SYMmetry

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An equivalence theorem concerning the three properties of \( E_n : S_n \rightarrow R = \left\{ -\infty, \infty \right\} \), namely, symmetry, recursivity and cyclic symmetry, is proved.

1. Introduction

Let for \( n = 2, 3, 4, \ldots \)

\[
\Gamma_n = \left\{ (p_1, p_2, \ldots, p_n) : p_i \geq 0, \ i = 1, 2, \ldots, n; \sum_{i=1}^n p_i = 1 \right\}
\]

denote the set of all finite \( n \)-component complete probability distributions with non-negative elements and \( S_n, (n = 2, 3, \ldots) \), the set of all \( 2n \)-tuples of the form \((p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)\) with \((p_1, p_2, \ldots, p_n) \in \Gamma_n, (q_1, q_2, \ldots, q_n) \in \Gamma_n\) such that \( p_i = 0 \) for all those indices \( i \) for which \( q_i = 0, 1 \leq i \leq n \). Let \( E_n : S_n \rightarrow R = \left\{ -\infty, \infty \right\} \), \( n = 2, 3, \ldots \) be a sequence of functions. We state below only those properties of \( E_n \) on which the whole discussion is based.

\( I_n \) (Recursivity) \( E_n : S_n \rightarrow R \) is recursive, that is, for all \((p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n\) with \( p_1 + p_2 > 0 \),

\[
E_n (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = E_{n-1} (p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, \ldots, q_n) + (p_1 + p_2) E_2
\]

\[
\left( \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} ; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right), \ p_1 + p_2 > 0.
\]

\( \ldots(1) \)

Notice that, in \( S_n, p_1 + p_2 > 0 \Rightarrow q_1 + q_2 > 0 \).

\( II_n \) (Symmetry) \( E_n : S_n \rightarrow R \) is symmetric, that is, for all
\((p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n,\)

\[ E_n (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = E_n (p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)}; q_{\pi(1)}, q_{\pi(2)}, \ldots, q_{\pi(n)}) \]  \(\ldots(2)\)

where \(\pi(1), \pi(2), \ldots, \pi(n)\) is an arbitrary permutation of \(1, 2, \ldots, n.\)

\(II_n^*\) (Cyclic Symmetry) \(E_n : S_n \rightarrow R\) is cyclic, that is, for all

\((p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n\)

\[ E_n (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = E_n (p_n, p_1, p_2, \ldots, p_{n-1}; q_n, q_1, q_2, \ldots, q_{n-1}). \] \(\ldots(3)\)

Obviously \(II_2\) and \(II_n^*\) are 'equivalent' and \(II_n^*\) \((n \geq 3)\) is weaker than \(II_n\) \((n \geq 3)\) in the 'strict sense'.

The object of this paper is to prove an equivalence theorem concerning the above mentioned three properties.

2. AN EQUIVALENCE THEOREM

The main result of this paper is the following:

**Theorem**—The following two systems are equivalent:

**System 1**: \(I_n\) \((n = 3, 4, \ldots)\) and \(II_n\) \((n = 2, 3, \ldots)\)

**System 2**: \(I_n\) \((n = 3, 4, \ldots)\) and \(II_n^*\) \((n = 3, 4, \ldots)\).

We first prove some lemmas.

**Lemma 1**—\(I_3\) and \(II_3^*\) imply

\[ E_2 (1, 0; 1,0) = E_2 (0,1; 0,1) = 0 \] \(\ldots(4)\)

\[ E_2 (p_1, p_2; q_1, q_2) = E_2 (p_2, p_1; q_2, q_1). \] \(\ldots(5)\)

**Proof**: Equation (4) is a simple consequence of \(I_3\) and \(II_3^*\). Hence its proof is omitted.
Let \((p_1, p_2; q_1, q_2) \in S_2\). If \(q_1 = 0\), then \(p_1 = 0\) and \(p_2 = q_2 = 1\); if \(q_1 = 1\), then \(q_2 = 0\) and consequently \(p_2 = 0, p_1 = 1\). In both these situations, (5) follows from (4). If \(0 < q_1 < 1\), then \(0 < q_2 < 1\). In this case

\[
E_2 (p_1, p_2; q_1, q_2) = \begin{cases} 
E_2 (1, 0; q_1, q_2) + E_2 (1, 0; 1, 0) & \text{if } p_2 = 0 \\
E_2 (1, 0; 1, 0) + E_2 (p_1, p_2; q_1, q_2) & \text{if } 0 < p_2 \leqslant 1 
\end{cases}
\]

\[
(1) \begin{cases} 
E_2 (0, 1; 0; q_1, 0, q_2) & \text{if } p_2 = 0 \\
E_2 (p_1, p_2; 0; q_1, q_2, 0) & \text{if } 0 < p_2 \leqslant 1 
\end{cases}
\]

\[
= \begin{cases} 
E_2 (0, 1; 0; q_2, q_1, 0) & \text{if } p_2 = 0 \text{ (using (3) once)} \\
E_2 (p_2, 0; p_1; q_2, 0, q_1) & \text{if } 0 < p_2 \leqslant 1 
\end{cases}
\]

(using (3) twice)

\[
(1) \begin{cases} 
E_2 (1, 0; 1, 0) + E_2 (0, 1; q_2, q_1) & \text{if } p_2 = 0 \\
E_2 (p_2, p_1; q_2, q_1) + p_2 E_2 (1, 0; 1, 0) & \text{if } 0 < p_2 \leqslant 1 
\end{cases}
\]

\[
(4) \quad = E_2 (p_2, p_1; q_2, q_1). 
\]

**Lemma 2—** \(I_n (n = 3, 4, \ldots)\) and \(I_n^* (n = 3, 4, \ldots)\) imply

\[
E_n (a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n; b_1, b_2, \ldots, b_{k-1}, 0, b_{k+1}, \ldots, b_n) = E_n (a_1, a_2, \ldots, 0, a_{k-1}, a_{k+1}, \ldots, a_n; b_1, b_2, \ldots, 0, b_{k-1}, b_{k+1}, \ldots, b_n) 
\]

\[
(6) \quad a_1 > 0; k = 2, 3, \ldots, n - 1; n = 3, 4, \ldots 
\]

\[
E_n (a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n) = E_n (a_2, a_1, a_3, \ldots, a_n; b_2, b_1, b_3, \ldots, b_n) 
\]

\[
(7) \quad 0 < a_1 + a_2 \leqslant 1; n = 3, 4, \ldots 
\]

\[
E_n (a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n) = E_n (a_1, a_3, a_2, \ldots, a_n; b_1, b_3, b_2, \ldots, b_n) 
\]

\[
(8) \quad a_1 > 0; n = 3, 4, \ldots 
\]

\[
E_n (a_1, a_2, a_3, \ldots, a_n; b_1, b_2, b_3, \ldots, b_n) = E_n (a_1, a_n-j+1, a_{n-j+2}, \ldots, a_n, a_2, a_3, \ldots, a_{n-j}; b_1, b_{n-j+1}, b_{n-j+2}, \ldots, b_n, b_2, b_3, \ldots, b_n) 
\]

\[
(9) \quad a_1 > 0; j = 1, 2, \ldots, n - 2; n = 3, 4, \ldots 
\]
PROOF: Equation (6) follows by repeated use of $I_n$ and (4); (7) from $I_n$ and (5); (8) can be proved by following the procedure explained in the proof of Proposition 2.3.5, pages 59-60 in Aczél and Daróczy¹; and (9) by using (3) and (7) alternatively depending upon the value of $j$. For instance, if $j = 1$, use (3) and (7) in this order only once.

**Lemma 3** — $I_n (n = 3, 4, \ldots)$ and $II^*_n (n = 3, 4, \ldots)$ imply

$$E_n (p_1, p_2, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n)$$

$$= E_n (p_2, p_3, \ldots, p_n; q_2, q_3, \ldots, q_n) \quad \text{for } n = 3, 4, \ldots \quad (10)$$

**PROOF:** Let $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$ where $n$ is any arbitrarily fixed integer $\geq 3$. If $0 < p_1 + p_2 < 1$, then (10) turns out to be (7). If $p_1 + p_2 = 0$, then $p_1 = p_2 = 0$. Let $p_d, 3 \leq d \leq n$, be the first positive element among $p_i$'s in $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$. Then $q_d > 0$ and have

$$E_n (p_1, p_2, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n) = E_n (0, 0, \ldots, p_d, \ldots, p_n; q_1, q_2, \ldots, q_d, \ldots, q_n) \quad (11)$$

If $q_1 = 0$ or $q_2 = 0$ or $q_1 = q_2 = 0$, then apply in succession to the right hand side of (11); (3) $n - d + 1$ times; then (6) to interchange $q_1$ and $q_2$; and finally (3) again but now only $d$ times. Then (10) follows.

If $0 < q_1 < 1, 0 < q_2 < 1$, then apply in succession to the right hand side of (11); (3) $n - d + 2$ times; (7); (9) with $j = d - 1$; (8); (9) with $j = n - d$; and finally (3) $d$ times. This proves (10).

**Proof of the main theorem**—System 1 obviously implies System 2. The fact that System 2 implies System 1 follows from $I_n (n = 3, 4, \ldots)$, (5), (10), and $II^*_n (n = 3, 4, \ldots)$ because (10) and $II^*_n (n = 3, 4, \ldots)$ imply $II_n (n = 2, 3, \ldots)$.

The proofs of Lemmas 2 and 3 are written descriptively to avoid writing lengthy mathematical calculations.

**Remarks (a)** — In the theory of permutation groups, it is known that the full permutation group $S_n$ can be generated by a cycle of length $n$ and a transport. Equation (3) gives rise to

$$E_n (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$$

$$= E_n (p_2, p_3, \ldots, p_n, p_1; q_2, q_3, \ldots, q_n, q_1)$$
which may be identified with the cycle $[1 \ 2 \ 3 \ ... \ n]$ of length $n$. Equation (10) may be identified with the transport [12], a cycle of length 2, indicating the interchange of $p_1$ with $p_2$ and of $q_1$ with $q_2$. This is the reason behind proving (10).

(b) Nath and Kaur$^4$ have already discussed the cyclic symmetry in connection with the Shannon entropy$^3$. In many axiomatic characterizations of symmetric information-theoretic measures like inaccuracy, directed divergence etc; properties such as continuity, symmetry and recursivity have been assumed. The equivalence theorem, proved above, says that in all such axiomatic characterizations, it is enough to assume recursivity and cyclic symmetry in place of recursivity and symmetry.

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References