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SOME EXISTENCE THEOREMS FOR OPERATOR EQUATIONS AND PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Some theorems for the existence of solution of a class of operator equations are given together with an application to a class of functional differential equations at resonance.

The method of alternative developed in recent years through the works of Cesari[2,3], Hale[8], Knobloch[9] and more recently in works of Locker, Kannan, Mawhin and others[6,10-12] with the help of functional analytic tools, has its roots in the classical bifurcation theory of Poincare, Lyapunov and Schmidt. The main idea consists of splitting of a given operator equation $Lx = Nx$ in a suitable space, into a system of auxiliary bifurction equations. These equations are then studied with any one of the variety of methods available in nonlinear analysis such as Banach's and Schauder's fixed point theorems, degree theory, monotone operator theory etc. In the application of the method of alternative to functional differential equations, for example, Mawhin[11], Das and Venkatesulu[7], the linear part is considered to be a linear ordinary differential operator. In Mawhin[11] the linear differential part consists of a simple first order differentiation operator where as Das and Venkatesulu[7] allow a more general form of differential operator. The present work deals with some abstract existence theorems and applications of functional differential equations.

Let C_T denote the space of continuous periodic functions of period T mapping the real line into R^n. Further suppose $C_{[-r,0]}$ denote the space of continuous functions from $[-r, 0]$ to R^n. By x_t, we denote a function in $C_{[-r,0]}$ defined as $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$. For simplicity, we shall assume that $T > r$. On thee space C_T and $C_{[-r,0]}$, the norms $\| \|_T$ and $\| \|_r$ denote the usual supremum norms. If $x \in C_T$ and $y \in C_T$ $\int_0^T x(s)\, y(s)\, ds = \langle x, y \rangle$ denote the usual inner product of the two elements x and y. It is clear $\| x \|_T \neq 0$ implies $\langle x, x \rangle > 0$. Let $f : R \times C_{[r,0]} \to R^n$ be a continuous function and A, an $n \times n$ matrix function of t. We shall apply one of the theorems developed here to a functional differential equation of the type.

$$\frac{dx}{dt} = A(t)\,x + f(t, x_t) \qquad \ldots(1)$$

and establish existence of periodic solutions.

Since the solutions are periodic functions, it is convenient to consider the above problem in the space of periodic functions C_T. Defining $Lx = \frac{dx}{dt} - Ax$ and $(Nx)(t) = f(t, x_t)$ for all $t \in [0, T]$, the equation (1) can be rewritten as

$$Lx = Nx \qquad \ldots(2)$$

with $D(L) \subset C_T$ and $D(N) \subset C_T$

$D(L)$ and $D(N)$ denoting the domain of L and N respectively. The operator L has the following properties[6,7] :

(a) $D(L)$ the space of all continuously differentiable functions in C_T is obviously dense in C_T

(b) L is a closed linear operator

(c) $R(L)$ is closed in C_T

(d) $C_T = R(L) + N(L^*)$, where L^* denotes the adjoint of L. Suppose that dim $N(L) = \dim N(L^*) = p$. Obviously, $p \leqslant n$. Let X_0 and Y_0 denote the p-dimensional subspaces $N(L)$ and $N(L^*)$ respectively. Further let P and Q denote suitable projections to be defined later from the space C_T to the subspaces X_0 and Y_0 respectively i.e. $R(P) = X_0$ and $R(Q) = Y_0$. Let $X_1 = R(I - P)$ and $Y_1 = R(I - Q)$. Thus the operator $L: D(L) \subset X_1 \to Y_1$ is bijection so that a partial inverse of L exists which is denoted by H. Further, H is a compact operator and the usual hypotheses requested by Cesari[5] are satisfied.

(i) $H(I - Q)\,Lx = (I - P)\,x \;\forall x \in D(L)$

(ii) $LPx = QLx \;\forall x \in D(L)$

(iii) $LH(I - Q) = I - Q$.

Since dim X_0 + dim $Y_0 = p \leqslant n$, there is an isomorphism between these two spaces, denoted by J say. By the wellknown procedure of Cesari[2] it is clear that eqn. (2) is equivalent to the two equations

$$\left.\begin{aligned} x &= Px + H(I - Q)\,Nx \\ QNx &= 0 \end{aligned}\right\}. \qquad \ldots(3)$$

Equations (3) can also be written as a single equation

$$x = Px + H(I - Q)Nx + JQNx. \qquad ...(4)$$

Thus the original problem of the existence of the solution of (1) has been reduced to the problem of existence of an abstract operator equation (4). Now we shall present some theorems concerning the existence of solutions of eqns. (1) and (4), and apply later for establishing periodic solution of an equation (12) of the form (1).

Main Results

Theorem 1—Let $N_\lambda(x)(t) = g(t, x_t, \lambda)$ be a continuous function from $[0, T] x C_{[r-0]} x [0, 1] \to R^n$ satisfying $N_0(x)(t) = f_0(t, x_t)$ and $N_1(x)(t) = f(t, x_t) = g(t, x_t, 1)$. Let D be a bounded open domain in C_T containing the origin. Suppose the following conditions hold :

(i) $\dot{x} = Ax. + \lambda g(t, x_t, \lambda)$ does not have a T-periodic solution on δD for each $\lambda \in (0, 1)$ where, δD denotes the boundary of D.

(ii) For $x \in \text{Ker } L \cap \delta D$, $QN_0 x \neq 0$

(iii) $d_{LS}(JQN_0, | \text{Ker } L, D \cap \text{Ker } L, 0) \neq 0$.

Then eqn. (1) has a T-periodic solution.

Note that d_{LS} stands for Leray-Schauder degree (see Berger and Berger[1]).

Proof : It is clear from (4) that $\dot{x} = Ax + \lambda g(t, x_t, \lambda)$ has T-periodic solution in δD if and only if

$$x = Px + \lambda H(I - Q)N_\lambda x + JQN_\lambda x \equiv F_\lambda(x) \qquad ..(5)$$

has a solution in δD. Thus (5) does not have a solution for all $\lambda \in (0, 1)$ (by condition (i)). Further, when $\lambda = 0$, the operator equation takes the form

$$x = Px + JQN_0 x. \qquad ...(6a)$$

Any periodic solution of (6a) obviously, satisfies $JQN_0 x = 0$ or equivalently, $QN_0 x = 0$. Further $x = Px$ and so $x \in \text{Ker } L$. Therefore by the conditions of the theorem the equation (6a) does not have a solution on δD.

So the Leray-Schauder degree $d_{LS}(I - F_1, D, 0)$

$$= d_{LS}(I - F_0, D, 0) = d_{LS}(I - F_0 | \text{Ker } L, \text{Ker } L \cap D, 0)$$

$$= d_{LS}(JQN_0 | \text{Ker } L, \text{Ker } L \cap D, 0) \neq 0.$$

Thus there is a solution of eqn. (1) which obviously is a periodic function.

The following three theorems deal with the abstract operator equation defined on C_T. However they are valid in more general spaces as well which we shall not pursue here.

The following theorem in a way generalizes a theorem of Cesari[5].

Theorem 2—Let $\Phi(\xi) \geqslant 0$ for $0 \leqslant \xi < \infty$ be a monotonic non-decreasing real valued function. Suppose $\|N(x)\| \leqslant \Phi(\|x\|)$ for all x. Further, let there exist constants R_0, C_0 such that for all $x \in C_T$ with $\|Px\| = R_0$, there is no $\theta \in R$ satisfying $JQNx = \theta Px$ and C_0 satisfiies the inequality $C_0 > R_0 + (M\chi' + q\chi)\Phi(C_0)$ where, $\|H\| \leqslant M$, $\|I - Q\| = \chi'$, $\|J\| = q$, $\|Q\| = \chi$. Then there is a periodic solution of the operator equation $Lx = Nx$.

PROOF: Let $D = \{x : \|Px\| \leqslant R_0 \text{ and } \|x\| \leqslant C_0\}$. Since under the usual hypotheses on L and N, as it was pointed out earlier, the equation $Lx = Nx$ is equivalent to eqn. (4), below we consider only eqn. (4).

Let $G_\lambda x \equiv \lambda(Px + H(I - Q)Nx + JQNx)$. Consider the map $(I - C_\lambda) : D \to C_T$. Suppose the equation (4) does not have a solution on δD, otherwise the theorem is complete, then $(I - G_1)x \neq 0$ for all $x \in \delta D$. Now we show that $(I - G_\lambda)x \neq 0$ for all $\lambda \in (0,1)$ and $x \in \delta D$. Suppose contrary, then there is $a\lambda_1 \in (0, 1)$ and an $x \in \delta D$ such that

$$\dot{x} = \lambda\ (Px + H(I - Q)Nx + JQNx) \qquad \ldots(6b)$$

is satisfied.

Suppose $\|Px\| \leqslant R_0$ and $\|x\| = C_0$, then from (6b), we get

$$C_0 = \|x\| \leqslant \lambda_1 [\|Px\| + (M\chi' + q\chi)\Phi(\|x\|)] \leqslant R_0 + (M\chi' + q\chi)\Phi(\|x\|)$$
$$\leqslant R_0 + (M\chi' + q\chi)\Phi(C_0) < C_0.$$

This is a cotradiction. Thus there is no $\lambda_1 \in (0, 1)$ with $\|Px\| \leqslant R_0$ and $\|x\| = C_0$ satisfying (6). In case $x \in \delta D$ in the sense that $\|Px\| = R_0$ and $\|x\| \leqslant C_0$, applying projection operator to both sides of eqn. (6b) we obtain

$$Px = \lambda_1 Px + \lambda_1 JQNx \qquad \ldots(7)$$

or

$$JQNx = \frac{(1 - \lambda_1)}{\lambda_1} Px.$$

But this is not possible by the hypothesis of the theorem and hence there is no λ_1 and $x \in \delta D$ for which eqn. (6) is satisfied. Thus $(I - G_\lambda)\, x \neq 0$ for all $\lambda \in (0, 1)$ and $x \in \delta D$.

Since in the case $\lambda = 0$, $(I - G_0)\, x \neq 0$ for $x \in \delta D$, we have

$$d_{LS}\,(I - G_1, D, 0) = d_{LS}\,(I - G_0, D, 0) = d_{LS}\,(I, D, 0) \neq 0.$$

Thus eqn. (4) has a solution. This complets the proof of the theorem.

Now we give a theorem where the conditions become somewhat more specific than in the above theorem and is realated to the Theorem 3.1 of Cesari[4].

Theorem 3—Let $\Phi(\xi) \geqslant 0$, $\Psi(\xi) \geqslant 0$ χ $0 \leqslant \xi < \infty$ be monotone nondecreasing functions with $\Psi(\xi) \geqslant Mx'(\xi)$. Suppose x also that $\|N(x)\| \leqslant \Phi(\|x\|)$ for all x. Let there exist $R_0 > 0$ such that $< J\,Q\,N\,x, P\,x > \leqslant 0$ [$< J\,Q\,N\,x, P\,x > \geqslant 0$] for all x with $\|P\,x\| = R_0$ and $\|(I - P)\,x\| \leqslant \Psi(\|x\|)$. Let there exist a $C_0 > R_0 + (Mx' + qx)\,\Phi(C_0)$. Then there is a solution of $Lx = Nx$.

PROOF : The proof runs in a way similar to the previous theorem. Let $D = \{x: \|P\,x\| \leqslant R_0$ and $\|x\| \leqslant C\}$. Assume $x = Px + H(I - Q)\,N\,x + JQNx$ does not have a solution on δD, else the theorem is proved. Now define $G_\lambda = \lambda\,(Px + H(I - Q)\,Nx + JQNx)$. We show that $(I - G_\lambda)\,(x) \neq 0$ for all $x \in \delta D$ and $\lambda \in (0, 1)$. Suppose contrary ,then $(I - G_\lambda)\,(x) = 0$ for some $\lambda_1 \in (0, 1)$ and $x \in \delta D$. Consider the case $\|Px\| \leqslant R_0$ and $\|x\| = C_0$. Then $C_0 = \|x\| \leqslant \lambda_1 \{\|Px\| + \|H(I - Q)\,N\,x\| + \|J\,Q\,N\,x\|\}$

$$\leqslant R_0 + (Mx' + qx)\,\Phi(C_0).$$

This contradicts a condition of the theorem. Now consider the case where $\|P\,x\| = R_0$ and $\|x\| \leqslant C_0$. Assume that

$$x = \lambda_1 P\,x + \lambda_1 H(I - Q)\,N\,x + \lambda_1 JQNx \qquad \ldots(8)$$

for some $\lambda_1 \in (0, 1)$ and $\|P\,x\| = R_0$ and $\|x\| \leqslant C_0$. Taking projection by P on bothh sides of (8) we have,

$$P\,x = \lambda_1 P\,x + \lambda_1 J\,QNx. \qquad \ldots(9)$$

Subtracting (9) from (8) we have

$$\|x - P\,x\| = \|\lambda_1 H(I - Q)\,N\,x\| \leqslant M\,x'\,\Phi(\|x\|) \leqslant \Psi(\|x\|)$$

Further,

$$\frac{1 - \lambda_1}{\lambda_1} P\,x = JQNx. \qquad \ldots(10)$$

Taking bracket operation on both sides of (10), we get

$$\frac{1-\lambda_1}{\lambda} < Px, Px > = < JQNx, P\,q > \leqslant 0.$$

But since $\|x\| \neq 0$, $< Px, Px > > 0$. This is a contradiction.

So $(I - G_\lambda)(x) \neq 0$ for all $\lambda \in [0, 1]$ and $x \in \delta D$.

Therefore, $d_{LS}(I - G_1, D, 0) = d_{LS}(I, D, 0) \neq 0$.

So $Lx = Nx$ has a solution in D.

In the next theorem a modified condition for the projection operator is assumed to solve the abstract equation (4).

Theorem 4—Let $\Phi(\xi) \geqslant 0$, $0 \leqslant \xi < \infty$ be a monotone nondecreasing function and $\|N(x)\| \leqslant \Phi(\|x\|)$ for all x. Suppose there exist real numbers R_0, C_0 such that

$C_0 > R_0 + (M x' + q x)\,\Phi(C_0)$ and there exist real numbers α, γ such that for all x for which $QN x = 0$,

$\|P x\| \leqslant \alpha \|(I - P) x\| + \gamma$ holds.

Further let

$R_0 > \alpha M \chi' \Phi(C_0) + \gamma$ and $d_{LS}(JQN \mid \text{Ker } L, D \cap \text{Ker } L, 0) \neq 0$.

Then there in a solution of the equation $Lx = Nx$.

Note that a condition of the type $\|Px\| \leqslant \alpha \|(I - P)\, x\| + \gamma$

when $QNx = 0$ was also requested by Mawhin[12].

PROOF : As in the proofs of earlier theorems, we shall consider the equivalent abstract equation

$$x = Px + H(I - Q)\, N x + JQNx.$$

Let

$$G_\lambda x = Px + \lambda H(I - Q)\, Nx + JQNx.$$

Let $D = \{x : \|P x\| \leqslant R_0 \text{ and } \|x\| \leqslant C_0\}$. We show first of all that $(I - G_\lambda)\, x \neq 0$ for all $\lambda \in (0, 1)$ and $x \in \delta D$. Assume contrary, then there is $\lambda_1 \in (0, 1)$ and an $x \in \delta D$ such that

$$x = G_{\lambda_1} x = P x + \lambda_1 H(I - Q)\, N x + JQNx \qquad \ldots(11)$$

In case $\|x\| = C_0$ and $\|Px\| \leqslant R_0$, we have from (11) $C_0 \leqslant R_0 + (M\chi' + P\chi)\,\Phi\ (C_0)$ which contradicts a condition of the theorem. Now assume that $\|Px\| = R_0$ and $\|x\| \leqslant C_0$. Operating both sides of (11) by P we have, $J\,QNx = 0 \Leftrightarrow QNx = 0$. So the equation (11) reduces to $(I - P\,x = \lambda_1\, H\,(I - Q)\,N\,x)$ and hence by a condition of the theorem $R_0 = \|Px\| \leqslant \alpha\,\|(I - P)\,x\| + \gamma \leqslant \alpha\, M\,\chi'\,\Phi\ (\|x\|) + \gamma \leqslant \alpha\, M\chi'\,\Phi\,(C_0) + \gamma$. But this is a contradiction since by a condition of the theorem $R_0 > \alpha\, M\,\iota'\,\Phi\,(C_0) + \gamma$. So $(I - G_\lambda)\,x \neq 0$ for all $\lambda \in (0, 1)$ and $x \in \delta\, D$. In the case where $\lambda = 0$, the $(I - G_0)\,x = 0$ reduces to $x = Px + JQNx$ which implies $x = Px$ and $QNx = 0$. But by the assumption $d_{LS}\ (JQN \mid \text{Ker } L,\ D \cap \text{Ker } L,\ 0) \neq 0$. So $d_{LS}\,(I - G_1,\ D,\ 0) = d_{LS}\ (JQN \mid \text{Ker } L,\ D \cap \text{Ker } L,\ 0) \neq 0$.

Therefore $Lx = Nx$ has a solution in D.

Now coming to the application of the theorem we present below an application of one of the theorems i.e. of Theorem 1 for a system of $2n$th order functional differential equations. For applications of Theorems 2 and 3 one can refer to the work of Cesari[4] to which these theorems are related.

Consider the following system of differential equations

$$y^{(2n)} + \frac{d}{dt}\ \text{grad}\ \Phi\,(y) + \mu \int_{-r}^{0} d\eta\,(\theta)\,y\,(t + \theta) = K\,(t) \qquad \ldots(12)$$

where y and k are n-dimensional vector functions and η is an $n \times n$ matrix function. Equation (12) can be rewritten as

$$y^{(1)} = z_1,\ z_1^{(1)} = z_2, \ldots, y_{2n+1}^{(1)} = -\frac{d}{dt}\ \text{grad}\ \Phi\,(y)$$

$$- \mu \int_{-r}^{0} d\,(\theta)\,y\,(t + \theta) + K\,(t). \qquad \ldots(13)$$

Now we formulate the following concerning the existence of periodic solutions of the above equation.

Theorem 5—Let K be a periodic function of period T with $\int_0^T K\,(s)\,ds = 0$.

Suppose η is a matrix function of bounded variation in $[-r, 0]$ and $M = \int_{-r}^{0} d\eta\,(\theta)$ is a nonsingular matrix. Suppose further that Φ is a twice continuously differentiable function, then the equation (12) has T-periodic solution for small μ.

PROOF : First of all we note that for each $u \in C_T$ and $v \in C_T'$ there is a constant C depending η_T such that

$$\int_0^T < \int_{-r}^0 d\eta(\theta)\, y(t+\theta)\, v(t) > dt \leqslant C \left(\int_0^T |u|^2\, dt\right)^{1/2} \left(\int_0^T |v|^2\, dt\right)^{1/2}.$$

In order to use the Theorem 1, we consider the auxilary equation

$$y_1^{(1)} = z_1, z_1^{(1)} = z_2, \ldots, z_{2n-1}^{(1)} = -\lambda \frac{d}{dt} \operatorname{grad} \Phi(y)$$

$$-\lambda \mu \int_{-r}^0 d\eta(\theta)\, y(t+\theta) + \lambda K(t) \qquad \ldots(14)$$

where $\lambda \in [0, 1]$.

Suppose $(\mathfrak{y}, \bar{z}_1, \ldots \bar{z}_{2n-1})$ is a T-periodic solution of (14).

Now integrating the equation (14) we have,

$$\int_0^T \left(\int_{-r}^0 d\eta(\theta)\, y(t+\theta)\right) dt = 0 = T \int_{-r}^0 d\eta(\theta) \left(\frac{1}{T} \int_0^T y(\theta)\, d\theta\right) = TMPY$$

where $Py = \frac{1}{T}\int_0^T y(\theta)\, d\theta$. Since H is nonsingular $Py = 0$. From equation (14) it is clear that y is a $2n$ times differentiable function with $\int_9^T y(\ )\, ds = 0$. Now making use of Parseval's relation we have

$$\int_0^T |y(t)|^2\, dt \leqslant \left(\frac{2\pi}{T}\right)^{2m} \int_0^T |y^{(m)}(t)|^2\, dt \qquad \ldots(15)$$

for all $m \leqslant 2n$ provided $y^{2n} \in L_2[0, T]$.

Noting that $\mathfrak{y}^{(1)} = \bar{z}_1, \ldots, \mathfrak{y}^{(2n-1)} = \bar{z}_{2n-1}$, we have from (14)

that

$$\mathfrak{y}^{2n} = -\lambda \frac{d}{dt} \operatorname{grad} \Phi(\mathfrak{y}) - \lambda \mu \int_{-r}^0 d\eta(\theta)\, \mathfrak{y}(t+\theta) + \lambda K(t). \qquad \ldots(16)$$

Now taking a scalar multiplication with $\bar{y}$ we get from equation (16)

$$\int_0^T \left(\bar{y}^{2n}, y \right) dt = -\lambda \int_0^T \left(\frac{d}{dt} \operatorname{grad} \Phi, \bar{y} \right) dt$$

$$-\lambda \mu \int_0^T \left(\int_{-r}^{0} d(\theta) \bar{y}(t+\theta), \bar{y}(t) \right) dt$$

$$+\lambda \int_0^T (K(t), \bar{y}(t))\, dt.$$

Integrating by parts and using periodicity of $\bar{y}$, we get

$$(-1)^n \int_0^T |\bar{y}^{(n)}|^2 dt = -\lambda \mu \int_0^T \left(\int_{-r}^{0} d\eta(\theta)\, \bar{y}(t+\theta), \bar{y}(t) \right) dt$$

$$+\lambda \int_0^T (K(s), \bar{y}(s))\, ds.$$

So $\int_0^T |y^{(n)}|^2 dt \leqslant \lambda \mu \subset \int_0^T |\bar{y}|^2 dt + \lambda \left(\int_0^T |k(s)|^2 ds\right)^{1/2} \left(\int_0^T |\bar{y}(s)|^2 ds\right)^{1/2}$.

Using the inequality (15) we have,

$$\int_0^T |y^{(n)}|^2 dt \leqslant \lambda \mu C \left(\frac{2\pi}{T} \right)^{2n} \int_0^T |y^{(n)}(t)|^2 dt$$

$$+\lambda \left(\frac{2\pi}{T} \right)^n \left(\int_0^T |k(s)|^2 ds \right)^{1/2} \left(\int_0^T |y^{(n)}(t)|^2 dt \right)^{1/2}.$$

If μ is small enough, so that $\mu C \left(\frac{2\pi}{T} \right)^{2n} < 1$, then $\int_0^T |y^{(n)}(t)|^2 dt \leqslant L$, where

L is independent of Φ and is dependent only on μ, η, T, K. So there is a constant L_1 such that $\|\bar{y}\|_\infty \leqslant L_1$, $\|\bar{z}_1\|_\infty \leqslant L$, ... $\|\bar{z}_{n-1}\|_\infty \leqslant L_1$.

Next, performing scalar multiplication with $\bar{y}^{(2n)}$ on both sides of eqn. (16) and integrating by parts, we have

$$\int_0^T | y^{(2n)}(t) |^2 \, dt \leqslant \lambda \left| \int_0^T \left(\frac{d}{dt} \text{ grad } \Phi(\mathfrak{y}), \mathfrak{y}^{(2n)} \right) dt \right|$$

$$+ \mu \lambda \int_0^T \left(\int_{-r}^0 d\eta(\theta) \mathfrak{y}(t+\theta), \mathfrak{y}^{2n}(t) \, dt \right|$$

$$+ \lambda \left| \int_0^T (K(s), y^{(2n)}(s)) \, ds \right| . \qquad \ldots(17)$$

Since $\|\mathfrak{y}\|_\infty \leqslant L_1$ and $\|\mathfrak{y}^{(1)}\|_\infty \leqslant L_1$, we have

$\left| \frac{d}{dt} \text{ grad } \Phi(\mathfrak{y}) \right| \leqslant L_2$ for some constant. Thus from (17) we have,

$$\int_0^T | \mathfrak{y}^{(2n)}(t) |^2 \, dt \leqslant L_2 T^{1/2} \left(\int_0^T | \mathfrak{y}^{(2n)} |^2 \, dt \right)^{1/2}$$

$$+ \mu C \left(\int_0^T | \mathfrak{y} |^2 \, dt \right)^{1/2} \left(\int_0^T | \mathfrak{y}^{(2n)} |^2 \, dt \right)^{1/2}$$

$$+ \left(\int_0^T | K(s) |^2 \, ds \right)^{1/2} \left(\int_0^T | \mathfrak{y}^{(2n)}(t) |^2 \, dt \right)^{1/2}.$$

So there is a constant L_3 such that

$\int_0^T | \mathfrak{y}^{(2n)}(t) |^2 \, dt \leqslant L_3$ where L_3 depends on μ, η, K, T and Φ only.

Now using the Parsevals relation, as in (15), we have a constant L_4 such that

$$\|\mathfrak{y}\|_\infty \leqslant L_4, \ \|z_1\|_\infty \leqslant L_4, \ \ldots \ \|\bar{z}_{2n-1}\| \leqslant L_4.$$

Choosing D in Theorem 1 as $D = [\{ (y, z_1, z_2 \ldots z_{2n-1}) :$

$$y \in C_T, z_1 \in C_T$$

and

$$\|y\|_\infty < 2L_4, \|z_1\|_\infty < 2L_4 \ (i = 1, \ldots 2n-1) \}$$

we see that eqn. (15) does not possess any periodic solution of period T in δD. In this case Ker $L = \{a, 0, 0, \ldots 0\}$ where a denotes and dimensional constant function.

Defining

$$Q(y_1, z_1, z_2, \ldots z_{2n-1}) = \left(\frac{1}{T}\int_0^T y(s)\, ds, 0, 0, \ldots 0\right),$$

we see that any $(a, 0, 0, \ldots 0) \in \text{Ker } L$

$$QN_0(a, 0, 0, \ldots 0) = (-\mu Ma, 0, 0, \ldots 0) \neq 0.$$

Further $d_{LS}(QN_0 x \mid \ker L, D \cap \ker L, 0) = \text{sing det } M \neq 0$. So all the conditions of Theorem 1 are satisfied and there is T-periodic solution of eqn. (12).

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