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COFLAT MODULES

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We know[19] that a right R-module C is flat if and only if every short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules is pure (in the sense of P. M. Cohn). Dually we define, in this paper, a right R-module A to be coflat if every short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules is copure. The notion of coflatness appears to be related to injectivity. Actually for a commutative noetherian ring R, coflat R-modules are injective. Coflat R-modules are precisely the absolutely pure modules (introduced and studied by Maddox[15] and Megibben[16]) when R is a commutative classical ring and for a commutative ring R, coflat R-modules are absolutely pure. Many properties of coflat modules are investigated over special types of rings.

Throughout this paper by a ring R we mean an associative ring with identity and by an R-module we mean an unitary right R-module.

Before defining a coflat module we recall :

Definitions 1—(i) An R-module M is said to be finitely embedded[20] (or cofinitely generated[12]) if $E(M) = E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_n)$ where $S_1, S_2, \ldots, S_n$ are simple R-modules (here $E(X)$ denotes the injective hull of an R-module X).

(ii) An R-module M is said to be cofree[8] if M is isomorphic to $\Pi \{E(S_\alpha) : S_\alpha$ is a simple R-module, $\alpha \in \Lambda\}$ for some index set Λ.

(iii) An R-module M is said to be cofinitely related[8] if there is an exact sequence $0 \to M \to N \to K \to 0$ of R-modules with N cofinitely generated, cofree and K cofinitely generated. (We remark that an R-module N is cofinitely generated, cofree if and only if N is injective with finitely generated,es sential socle (Hiremath[8], p. 3; Lemma 1 of Vamos[20]).

(iv) A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is said to be copure[9] if every cofinitely related R-module is injective relative to this sequence.

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We recall [Proposition 4 of Stenström[19]] that an R-module C is flat if and only if every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is pure. Using this characterization of flat modules we dually define

Definition 2—An R-module A is said to be coflat if every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is copure.

It is an easy observation that an R-module A is coflat if and only if A is a copure submodule of every R-module in which it is embedded. So a coflat module can also be called as an absolutely copure module.

Since copure short exact sequences of R-modules form a 'Proper Class' [in the sense of MacLane[14] (p. 367)] (Proposition 8 of Hiremath[9])], it follows that an R-module is coflat if and only if it is copure in its injective hull.

Proposition 3—For a ring R the following conditions are equivalent.

(i) Every R-module is coflat;

(ii) every short exact sequence of R-modules is copure;

(iii) every cofinitely related R-module is injective;

(iv) R is a right V-ring.

PROOF : (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from [Proposition 5 of Hiremath[9]] and (i) $\Leftrightarrow$ (ii) are obvious.

Remark 4 : Clearly every injective R-module is coflat but not conversely.

Example—Let $R = \prod_{p \in P} Z/pZ$ where P is the set of all positive primes. Since R is a commutative Von Neumann regular ring, it is a V-ring by (Rosenberg[18], Theorem 6). Then, by the above proposition, every R-module, in particular the ideal $S = \bigoplus_{p \in P} Z/pZ$ of R, is coflat. But S is not injective as S is a proper essential ideal of R.

As an easy consequence of the definition we get the following dual of the well-known result that a finitely presented flat module is projective (Faith[7], Corollary 11.30).

Proposition 5—(i) Every cofinitely related coflat R-module is injective (ii) every copure submodule (in particular a direct summand) of a coflat R-module is coflat.

PROOF : (i) follows from the facts that a coflat module is copure in its injective hull and a cofinitely related copure submodule is a direct summand [Hiremath[9], Proposition 4 (ii)].

(ii) follows from the facts that an R-module is coflat if and only if it is copure in its injective hull and copure short exact sequences of R-modules form a proper class.

Proposition 6—Coflats are closed under taking extensions and finite direct sums.

PROOF : $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules with A, C coflat. Without loss of generality we can assume that A is a submodule of B and $C = B/A$ and the maps are canonical. Since A, C are coflat, A is copure in $E(B)$ and the C is copure in $E(B)/A$. Since copure short exact sequences of R-modules form a proper class it follows that B is copure in $E(B)$ and thus proving the coflatness of B.

The second assertion follows from the first and finite induction.

We do not know whether for an arbitrary ring R, every direct sum of coflat (even injective) R-modules is coflat.

Proposition 7—Every direct sum of coflat R-modules is coflat if and only if every direct sum of injective R-modules is coflat.

PROOF : Only if it is obvious.

If Let $\{A_I\}_{-II}$ be any family of coflat R-modules. Since A_I is copure in $E(A_I)$ for each $i \in I$, it follows, from[Hiremath[9], Proposition 9], that $\underset{i \in I}{\oplus} A_I$ is copure in $\underset{i \in I}{\oplus} E(A_I)$. Now the latter module is coflat by hypothesis and so is the former by Proposition 5 (ii).

Since, by Faith[6] (p. 53), every direct sum of injective modules over a right noetherian ring is injective, we have

Corollary 8—If R is a right noetherian ring then every direct sum of coflat R-modules is coflat.

Proposition 9—If R is a right hereditary ring then every direct sum of injective R-modules is coflat.

PROOF : Let $A = \underset{i \in I}{\oplus} A_I$ where each A_I is an injective R-module. Let $\alpha : A \to M$ be any homomorphism of A into a cofinitely related R-modules M. Let $a_1,\ldots,a_n \in A$ such that $\alpha(a_1),\ldots,\alpha(a_n)$ generate the socle of Im (α). Let $i_1,\ldots,i_k \in I$ such that $a_j \in A_{I_1} \oplus \ldots \oplus A_{I_k} = B$ for each $j = 1,\ldots,n$. Since socle of Im (α) is essential in Im (α) and socle of Im (α) is contained in $\alpha(B)$, it follows that $\alpha(B)$ is essential in Im (α). Since R is right hereditary and B is injective it follows from Theorem I. 5.4 of Cartan and Eilenberg[3] that $\alpha(B)$ is injective. Thus $\alpha(B) =$ Im (α)

and hence Im (α) is injective. So α extends to a homomorphism β of $E(A)$ into M. This proves the copurity of A in $E(A)$ and hence the coflatness of A.

Corollary 10—If R is a right hereditary ring then

(i) every direct sum coflat R-modules is coflat

(ii) every factor module of a coflat R-module is coflat.

PROOF : (i) follows from Propositions 7 and 9.

(ii) Let M be a coflat R-module and let N be a submodule of M. Since M is coflat, M is copure in $E(M)$ and hence, M/N is copure in $E(M)/N$ since copure short exact sequences of R-modules form a proper class. Since R is right hereditary, $E(M)/N$ is injective by Theorem I. 5.4. of Cartan and Eilenberg[3]. Now the coflatnes of M/N follows from Proposition 5 (ii).

Remark 11 : The converse of (ii) of the above corollary is not true. In fact there exists a ring R not even right semi-hereditary for which (ii) of the above corollary is true.

Example—Cozzens and Johnson[4] have constructed, for each positive integer n, a right neotherian, a right V-domain R of right global dimension n. If we take $n=2$, then we have a right noetherian right V-domain R of right global dimension 2 which is clearly not hereditary and hence not semi-hereditary as R is noetherian. Since R is a V-ring, (ii) of Corollary 10 is true by Proposition 3.

Remark 12 : We know[6] (p. 53) that a ring R is right noetherian if and only if every direct sum of injective R-modules is injective. But this result is no longer true if 'injective' is replaced by 'coflat'.

Example—Because of Corollary 10 (i) we need only find a right hereditary ring which is not right noetherian. Let X be a countably infinite discrete space and let X^* be the one-point compactification of X. Let k be a field with discrete topology. Let R be the ring of all continuous functions from X^* into k. Let $X = \{x_1, x_2, \ldots x_n, \ldots\}$. For $n \in N$, $a_n = \{f \in R : f(x_i) = 0 \text{ for each } i \geqslant n\}$. It is an easy verification that each a_n is an ideal of R and $a_{1+} \subset a_{2+} \subset \ldots a_{n+} \subset a_{n+1+} \subset \ldots\ldots$ Thus R is a commutative non-noetherian ring. But by Corollary 4.6 and Example 5.1 of Bergman[1] R is a hereditary ring.

We do not know whether every direct product of coflat modules is coflat.

We recall that (i) an R-module A is said to be absolutely pure[15,16] if it is a pure submodule of every module in which it is contained.

(ii) an R-module A is said to be (Damiano) coflat (Definition 1.2 of Damiano[5]) (which we call D-coflat) if, for each positive integer n and for each $a \in R^n$, if $x \in M^n \setminus Ma$ (where $Ma = \{(ma_1,..., ma_n) : m \in M\}$ with $a = (a_1...,a_n)$), then $xC \neq 0$ for some $n \times m$ matrix C over R such that $aC = 0$.

Proposition 13—(i) Over a commutative ring R, every coflat R-module is absolutely pure.

(ii) Over a commutative classical ring[21] R, an R-module is absolutely pure if and only if it is coflat.

PROOF : Follows from the fact (Proposition 13 of Hiremath[10]) that over a commutative ring, copurity implies purity and the fact (Proposition 13 of Hiremath[10] and Corollary 16 of Hiremath[11]) that copurity and purity equivalent for a commutative classical ring.

Corollary 14—A commutative classical ring R is noetherian if and only if every coflat R-module is injective.

PROOF : Follows from Proposition 13 and Theorem 3 of Megibben[16].

We also have, from Proposition 13 and Corollary 2 of Megibben[16].

Proposition 15—Every coflat R-module, for a commutative ring R, is injective relative to every short exact sequence $0 \to P' \to P \to K \to 0$ of R-modules, with P projective and P' finitely generated. The converse is true if R is commutative classical.

Corollary 16 – If R is a commutative classical ring then

(i) any direct product of coflat R-modules is coflat;

(ii) any direct sum of coflat R-modules is coflat.

PROOF : (i) follows from Proposition 15.

(ii) follows from Proposition 13 and Proposition 2 of Megibben[16].

We know [Faith[7], 11.25.2] that every flat R-module, over an integral domain R, is torsion-free. We have the dual result when R is commutative.

Proposition 17—If R is a commutative integral domain then every coflat R-module is divisible.

PROOF : Follows from Proposition 13 (i) and [Megibben[16], p. 565]

We thus have the following implications.

$$\text{injective} \overset{\text{(i)}}{\Rightarrow} \text{coflat} \overset{\text{(ii)}}{\Rightarrow} \text{absolutely pure} \overset{\text{(iii)}}{\Rightarrow} D\text{-coflat} \overset{\text{(iv)}}{\Rightarrow} \text{divisible},$$

where (i) and (iii) are true in general (Remarks 4 and Proposition 1.14 of Damiano[5]], (ii) holds for commutative rings (Proposition 13 (1)) and (iv) for commutative integral domains (Proposition 1.6 of Damiano[5]).

We do not know whether, for a commutative integral domain R, every divisible R-module is coflat. However, we have, from Proposition 13 and Theorem 6 of Megibben[16] :

Theorem 18—If R is a commutative classical domain then R is a Prüfer domain if and only if every divisible R-module is coflat.

Theorem 19—If R is a commutative classical domain then R is a Prüfer domain if and only if every homomorphic image of a coflat R-module if coflat.

PROOF : Follows from Proposition 13 and Theorem 2 of Megibben[16].

We know (Bourbaki[2], I 2.5, Proposition 5) that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and if C is flat and then A is flat if and only if B is flat. We now consider the dual problem : if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and A is coflat, then is it true that B is coflat if and only if C is coflat ? The 'if' part is answered in the affirmative in Proposition 6. Though we are not able to answer the 'only if', in general, we have the affirmative answer in some special cases.

Proposition 20—Let $0 \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules, where R is a commutative integral domain. If A, B are coflat and $t(B)$ (the torsion part of B) is contained in Im (f) then C is injective (and hence coflat).

PROOF : Without loss of generality we assume that A is a submodule of B, $C = B/A$ and the maps are the canonical maps. Since A is coflat, A is copure in B and hence A is pure in B (Proposition 13 of Hiremath[10]) which in turn implies that A is w-pure[9] (or relatively divisible[22]) in B. Since $t(B) \subseteq A$, it now follows that C is torsionfree. Since B is divisible, by Proposition 17, C is also divisible. Now C is injective by [Cartan and Eilenberg[3], Proposition VII.1.3].

Remark 21 : It is well-known[13] that an R-module A is flat if and only if it is the direct limit of a directed system of projective R-modules. The dual result is not true for coflat modules. Actually neither the inverse limit of an inverse system of injective R-modules is coflat [(see example (i) below)] nor can every coflat R-module be expressed as the inverse limit of an inverse system of injective R-modules [example (ii)].

Examples—(i) $A = \bigoplus_{n=1}^{\infty} D_n$ where $D_n = Z(p^\infty)$ for each n, for a fixed prime p.

Let $D_n = < a_{ni} / pa_{n1} = 0,\ a_{ni} = pa_{n,i+1};\ i = 1, 2\ldots >$

for $n = 1, 2,\ldots$

Define $E_n = \bigoplus_{m=n+1}^{\infty} D_m$, $n = 1, 2, \ldots$

Define $S_1 = A$. Let $E(< a_{12} + a_{21} >)$ be the divisible hull of $a_{12} + a_{21}$ in $D_1 \oplus D_2$ (note that $a_{12} + a_{21}$ $D_2 \oplus D_2$ which is divisible). Define $S_2 = E(< a_{12} + a_{21} >) \oplus E_2$. Now let $E(< a_{12} + a_{21} + a_{31} >)$ be the divisible hull of $a_{12} + a_{21} + a_{31}$ in S_2. Define $S_3 = E(< a_{12} + a_{21} + a_{31} >) \oplus E_3$. In this way we have a strictly descending chain $S_1 \supsetneq S_2 \supsetneq \ldots\ldots$ of subgroups of A where $S_n = E(< a_{12} + a_{21} + \ldots + a_{n1} >) \oplus E_n$ for $n = 2, 3, \ldots\ldots$ with $S = \bigcap_{n=1}^{\infty} S_n = < a_{11} >$ by observing that $\bigcap_{n=2}^{\infty} E(< a_{12} + a_{21} + \ldots + a_{n1} >) = < a_{11} >$. Now $\{S_n, \pi_m^n, n \geqslant m\}$ is an inverse system of divisible (= injective) abelian groups, where each π_m^n $S_n \to S_m$ is inclusion map, with $\{S, \pi_n\}$ ($\pi_n : S \to S_n$ is the inclusion map for each n) as its inverse limit. Since S is not divisible (= injective), it is not coflat (as a Z-module) by Corollary 14.

(ii) Let $R = \prod_{p \in P} Z/pZ$ where P is the set of all positive primes. Let $S = \bigoplus_{p \in P} Z/pZ$. Then R is the injective hull of S as an R-module and every epimorphic image of the semisimple R-module S is of the form $\bigoplus_{p \in P} Z/pZ$ where P' is a subset of P and its injective hull will be of the form $\prod_{p \in P'} Z/pZ$.

Since R is a V-ring, every R-module, in particular S, is coflat. Suppose S is the inverse limit of an inverse system of injective R-modules, say $S = \varprojlim \{A_i, \pi_j^i, i \geqslant j, i, j \in I\}$ where A_i's are injective. Let $\pi_i : S \to A_i$ be the canonical morphism for each $i \in I$. Since A_i contains the injective hull of $\pi_i(S)$ which will be a direct summand of A_i we may as well assume that $S = \varprojlim \{A_i, \pi_j^i \; i \geqslant j, i, j \in I\}$ where A_i is the injective hull of $\pi_i(S)$. By the remark made above, each A_i will be of the form $\prod_{p \in P'} Z/pZ$ where P' is a subset of P. So each A_i is complete in the n-adic topology. Since S is the inverse limit of the inverses ystem $\{A_i\}$, S will be complete in the n-adic topology which implies, by Proposition 2 of Rangaswamy[17], that S is a direct summand of R, a contradiction. So S is not the inverse limit of an inverse system of injective R-modules.

Remark 22 : We recall that the Schanuell's lemma for injective modules states that if $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C' \to 0$ are short exact sequences of R-modules, with B, B' injective, then $B \oplus C'$ and $B' \oplus C$ are isomorphic. Since there exist coflat modules which are not injective (cf. Remark 4) the coflat analogue of Schanuell's lemma is not true. So we cannot speak of coflat dimension of an R-module.

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