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UNSTEADY SELF-SIMILAR STAGNATION POINT BOUNDARY LAYERS FOR MICROPOLAR FLUIDS

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The self-similar solution of the unsteady laminar incompressible two-dimensional and axisymmetric stagnation point boundary layers for micropolar fluids governing the flow and heat transfer problem has been obtained when the free stream velocity and the square of the mass transfer vary inversely as a linear function of time. The nonlinear ordinary differential equations governing the flow have been solved numerically using a quasilinear finite-difference scheme. The results indicate that the coupling parameter, mass transfer and unsteadiness in the free stream velocity strongly affect the skin friction, microrotation gradient and heat transfer whereas the effect of microrotation parameter is strong only on the microrotation gradient. The heat transfer is strongly dependent on the Prandtl number whereas the skin friction gradient are unaffected by it.

1. INTRODUCTION

The theory of micropolar fluid including thermomicropolar fluid was developed by Eringen[1-3]. This theory includes the effects of micro-inertia and couple stresses and can explain the non-Newtonian behaviour of certain polymeric fluid, animal blood, and liquid crystals. The concept of boundary-layer theory in micropolar fluid was first introduced by Willson[4]. Since then, several authors [5-9] have studied the steady self-similar flow of micropolar fluids for two dimensional bodies with or without pressure gradient. It is well known that most problems encountered in applications are unsteady. The unsteady flow and heat transfer problem in micropolar fluid has not been studied so far.

The objective of this paper is to find self-similar solution of the unsteady incompressible boundary layer flow of micropolar fluids with mass transfer at the stangnation point of a two-dimensional and an axisymmetric body (cylinder and sphere). The differential equations governing the flow have been solved numerically using a quasilinear finite-difference scheme. The results have been compared with those available in literature[6,8,9].

2. Governing Equations

We consider the unsteady incompressible laminar boundary-layer flow of micropolar fluid in the neighbourhood of the stagnation point of a two-dimensional and an axisymmetric body when the free stream velocity and square of the mass transfer vary inversely as a linear function of time. The free stream fluid temperature is assumed to be constant and the dissipation parameter is considered to be negligible in the neighbourhood of the stagnation point. Under the above assumptions, the unsteady boundary layer equations with three independent variables (x, y, t) governing the flow of micropolar fluids at the stagnation point of a two-dimensional and an axisymmetric body given by[1-3]

$$(r^j u)_x + (r^j v)_y = 0 \qquad \ldots(1a)$$

$$u_t + uu_x + vu_y = u_e (u_e)_x + (u_e)_t + [(\mu + k_1)/\rho]\, u_{yy} + (k_1/\rho)\, N_y \qquad \ldots(1b)$$

$$N_t + uN_x + vN_y = G_1 N_{yy} - \sigma_1 (2N + u_y) \qquad \ldots(1c)$$

$$T_t + uT_x + vT_y = Pr^{-1} (\mu/\rho)\, T_{yy} + (\alpha_1/\rho c_p)\, (T_x N_y - T_y N_x) \qquad \ldots(1d)$$

can be reduced to ordinary differential equations with η as an independent variable. These equations, which govern the self-similar flow, can be written in dimensionless from as

$$(1 + K) F'' + [f - (1 + j)^{-1} \lambda \eta/2] F' + (1 + j)^{-1} (1 - F^2) + (1 + j)^{-1} \lambda (1 - F) + Ks' = 0 \qquad \ldots(2a)$$

$$Gs'' + [f - (1 + j)^{-1} \lambda\eta/2]s' - (1 + j)^{-1} sF - (3/2) \lambda (1 + j)^{-1} s - 2 (1 + j)^{-1} \sigma s - (1 + j)^{-1} \sigma F' = 0 \qquad \ldots(2b)$$

$$Pr^{-1} g'' + [f - \alpha s - (1 + j)^{-1} \lambda\eta/2] g' = 0. \qquad \ldots(2c)$$

The boundary conditions are

$$\begin{aligned} &F = 0,\ s = -F_w'/2,\ g = 1 \text{ at } \eta = 0 \\ &F \to 1,\ s \to 0,\ g \to 0 \text{ as } \eta \to \infty \end{aligned} \qquad \ldots(3)$$

where

$$\begin{aligned} &\eta = (1 + j)^{1/2} (a\rho/\mu)^{1/2}\ (1 - \lambda t^*)^{-1/2}\, y,\ t^* = at \\ &u_e = ax\ (1 - \lambda t^*)^{-1},\ u = u_e F,\ F = f' \end{aligned} \qquad \ldots(4a)$$

$$a = (du_e/dx)_{t^*=0},\ \lambda t^* < 1$$

$$v = -(1+j)^{1/2}\ (a\mu/\rho)^{1/2}\ (1-\lambda t^*)^{-1/2}\ f$$
$$N = a^{3/2}\ (1+j)^{1/2}\ (\rho/\mu)^{1/2}\ x\ (1-\lambda t^*)^{3/2}\ s \qquad \ldots(4b)$$

$$(T - T_\infty)/(T_w - T_\infty) = g,\quad G_1 = \gamma/\rho\lambda_1,\ \sigma_1 = k_1/\rho\lambda_1$$
$$K = k_1/\mu,\ G = G_1\,\rho/\mu,\quad \sigma = (\sigma_1/a)\,(1-\lambda t^*) \qquad \ldots(4c)$$

$$\alpha = (\alpha_1 a/\mu c_p)\ (1-\lambda t^*)^{-1},\quad f = \int_0^\eta F\,d\eta + f_w$$
$$f_w = -(1-j)^{1/2}\ (Re_x)^{1/2}\ (v_w/u_e)_{t^*=0}\,(1-\lambda t^*)^{1/2} \qquad (4d)$$
$$Re_x = (u_e)_{t^*=0}\ x/(\mu/\rho),\ (u_e)_{t^*=0} = ax.$$

Here x and y are the distances along and perpendicular to the surface, respectively; t and t^* are the dimensional and dimensionless times, respectively; u and v are the dimensional velocity components in the x and y directions, respectively; F (or f') is the dimensionless velocity in the x direction; N and s are the dimensional and dimensionless microrotation components, respectively; T and g are the dimensional and dimensionless temperatures, respectively; r is the radius of revolution of an axisymmetric body; $j = 0$ for a two-dimensional body and $j = 1$ for an axisymmetric body; a is the gradient of velocity at the edge of boundary layer at $t^* = 0$; F'_w is the velocity gradient at the wall; Pr is the Prandtl number;, ρ and α_1 are the density and micropolar heat conduction coefficients, respectively: c_p is the specific heat at a constant pressure; and μ, k_1, γ an λ_1 are, respectively, the viscosity, vortex viscosity, spin-gradient viscosity and micro-inertia density. G_1 and σ_1 are the parameters defined in (4c) and Re_x is the local Reynolds number. K and σ represent coupling parameters; G and α, respectively, represent microrotation and micropolar heat conduction parameters; and the prime denotes derivatives with respect to η. The parameters α and σ are constants if α_1 and σ_1 are assumed to vary as $(1-\lambda t^*)$ and $(1-\lambda t^*)^{-1}$, respectively. The subscripts t, t^*, x and y denote derivatives with respect to t, t^*, x and y, respectively; and the subscript e denotes conditions at the edge of the boundary layer. If the normal velocity at the wall $v_w/(u_e)_{t^*=0}$ varies as $(1-\lambda t^*)^{-1/2}$ and Re_x is taken as a constant, then the mass transfer parameter f_w will be a constant. $f_w \gtrless 0$ according to whether it is suction or injection. It is assumed that the injected fluid possesses the same physical properties as the boundary layer fluid and has a static temperature equal to the wall temperature. Also, λ is the parameter denoting the unsteadiness in the free stream velocity and $\lambda > 0$ for accelerating flows and $\lambda < 0$ for decelerating flows.

The steady-state equations ($\lambda = 0$) represented by (2a – c) for $j = 0$ (two-dimensional case) and $\alpha = 0$ (i. e., in the absence of micropolar heat conduction) are the same as those of Gorla[9]. Also, eqns. (2a) and (2b) reduce to those of Nath[6] if we

replace $(1 + K)$ by 1 and $(1 + j)^{-1}$ by β in (2a) and put $fs' - (1 + j)^{-1/2} sF = 0$ in (2b). It may be noted that the temperature field was not considered by Nath[6]. Our boundary conditions given by eqn. (3) reduce to those of Nath[6] and Gorla[9] if we put $s = 0$ and $\eta = 0$. Furthermore, the steady-state eqns. (2a — c) with $\lambda = 0$ for $j = 0$ are also identical with those of Ramachandran and Mathur[8] if we put $K = \sigma = N_1$, $G = N_3$ and replace 1 by N_2 in the coefficient of $(fs' - sF)$. Also the boundary conditions at $\eta = 0$ given by $s = -2^{-1} F_w'$ has to be replaced by $s = -(\lambda^2/2) F_w'$ where λ_2 is a parameter having range $0 \leqslant \lambda_2 \leqslant 1$. However, eqn. (2c) is slightly different from Ramachandran and Mathur[8] as they have taken parabolic distribution of wall temperature. It may be remarked that for $K = \alpha = 0$, eqns. (2a) and (2c) reduce to those of classical Newtonian fluid flow at the stagnation point and eqn. (2b) becomes superfluous. It may be noted that we have used the boundary condition $s = -F_w'/2$ at $\eta = 0$ instead of $s = 0$ (see eqn. (3)). In other words, we have assumed that the microrotation at the wall is equal to the angular velocity. The justification for using such a boundary condition is given in detail in Ahmadi[7], and Ariman *et al.*[7].

The skin-friction coefficient at the wall can be expressed as

$$C_f = 2\tau_w/\rho u_e^2 = 2(1 + j)^{1/2} (\bar{R}e_x)^{-1/2} [(1 + K) F_w' + Ks_w] \quad ...(5a)$$

where

$$\tau_w = [(\mu + k_1) (\partial u/\partial y) + k_1 N]_{y=0}, \ \bar{R}e_x = (u_e x \rho)/\mu. \quad ...(5b)$$

The heat-transfer coefficient in terms of Nusselt is given by

$$Nu = (xq_w)/K_c (T_w - T_\infty) = -(1 + j)^{1/2} (\bar{R}e_x)^{1/2} (g_w' + \beta s_w) \quad ...(6a)$$

where

$$q_w = -[K_c (\partial T/\partial y) + \beta_1 (\partial N/\partial x)]_{y=0}, \ \beta = \beta_1 a/[K_c (T_w - T_\infty)]. \quad ...(6b)$$

The couple stress coefficient is expressed in the form

$$M = m_w/(\rho u_e^2) = (1 + j) G s_w' \quad ...(7a)$$

where

$$m_w = (\gamma x/\lambda_1) (N_y)_{y=0}. \quad ...(7b)$$

Here τ_w, q_w and m_w are, respectively, the stress, heat-transfer rate and couple stress at the wall; C_f, Nu and M are, respectively, the skin-friction coefficient, Nusselt

number and couple stress coefficient; F'_w , s'_w and g'_w are, respectively, the skin-friction, microrotation gradient and heat transfer parameters; β and β_1 are the dimensionless and dimensional heat conduction parameters, respectively; K_c is the thermal conductivity; and $\bar{R}e_x$ is the local Reynolds number.

3. Results and Discussion

Equations (2a-c) under boundary conditions (3) have been solved numerically using a quasilinear finite-difference scheme which is described in detail in Bellman and Kalaba[12] and Inouye and Tate[13]. Hence, the description of the method is not presented here. Computations have been carried out for various values of the parameters λ, K, G, Pr and f_w with $\sigma = 1.5$, $\alpha = 1$ and $\beta = 0.3$. The effect of step size $\Delta\eta$ and the edge of the boundary layer η_∞ on the solution has been studied with a view to optimize them. Finally, the computations were carried out with $\Delta\eta = 0.02$ and η_∞ has been taken between 4 to 12 depending on the values of the parameters. The results presented here are found to be independent of the step size $\Delta\eta$ and the edge of the boundary layer η_∞ at least up to 4th decimal place.

In order to assess the accuracy of our method, we have compared our steady state ($\lambda = 0$) results for skin-friction parameter F'_w and gradient of microrotation s'_w with those of Nath[6] and Ramachandran and Mathur[8] and heat-transfer parameter $\left(-g'_w\right)$ with those of Gorla[9] and found them in excellent agreement. The comparison is shown in Tables I–III.

Table I

Comparison of skin-fraction and microrotation gradient parameters F'_w and s'_w with those of Nath[6] for $t^* = j = A = 0$

	F'_w			F'_p		
$G \diagdown K$	0.01	0.1	1.0	0.01	0.1	1.0
1.5	1.2325	1.2254	1.1527	0.3688	0.3638	0.3688
	(1.2318)	(1.2247)	(1.1520)	(0.3688)	(0.3688)	(0.3687)
2.5	1.2326	1.2274	1.1734	0.2473	0.2473	0.2472
	(1.2320)	(1.2267)	(1.1727)	(0.2473)	(0.2473)	(0.2472)
3.5	1.2328	1.2285	1.1848	0.1882	0.1882	0.1882
	(1.2321)	(1.2278)	(1.1842)	(0.1882)	(0.1882)	(0.1882)
4.5	1.2328	1.2292	1.1923	0.1528	0.1528	0.1528
	(1.2322)	(1.2285)	(1.1917)	(0.1527)	(0.1527)	(0.1527)

The values in parentheses are tabulated by Nath[6]

TABLE II

Comparison of skin-friction, microrotation gradient and microtation at the wall F_w', s_w' *and* s_w *with those of Ramachandran and Mathur*[8] *for* $t^* = 0$, $G = 13.5$ $\sigma = 4.5$, $N_2 = 9$, $Pr = \alpha = 1$ *and* $K = 4.5$.

λ_2	F_w'	s_w'	s_w
0	0.4945	—0.1218	0
	(0.4972)	(−0.1238)	0
0.02	0.4975	—0.1172	—0.0050
	(0.5001)	(−0.1194)	(—0.0050)
0.5	0.5817	0.0057	—0.1454
	(0.5834)	(0.0068)	(—0.1458)
0.9	0.6773	0.1574	—0.3047
	(0.6774)	(0.1531)	(−0.3048)
1.0	0.7064	0.2024	—0.3532
	(0.7058)	(0.1981)	(—0.3529)

The values in the parentheses are tabulated by Ramachandran and Mathur[8].

TABLE III

Comparison of heat transfer parameter $-g_w'$ *with those of Gorla*[9] *for* $t^* = j = A = 0$ *and* $Pr = 5$

G	σ	$K = 0.5$	$K = 1.5$	$K = 4.5$	$K = 13.5$	$K = 40.5$
1.5	0.5	0.9856	0.9149	0.8125	0.6996	0.5952
		(0.9853)	(0.9146)	(0.8122)	(0.6994)	(0.5937)
0.5	1.5	0.9877	0.9198	0.8221	0.7139	0.6117
		(0.9873)	(0.9195)	(0.0218)	(0.7137)	(0.6109)
13.5	1.5	0.9853	0.9138	0.8094	0.6932	0.5848
		(0.9851)	(0.9137)	(0.8093)	(0.6930)	(0.5839)
40.5	1 5	0.9859	0.9151	0.8116	0.6953	0.5867
		(0.9857)	(0.9149)	(0.8114)	(0.6953)	(0.5846)

The values in the parentheses are tabulated by Gorla[9].

The results for the two dimensional flow ($j = 0$) are presented in Figs. 1-5 and for the axisymmetric flow ($j = 1$) in Figs. 6-8. The effects of the coupling parameter

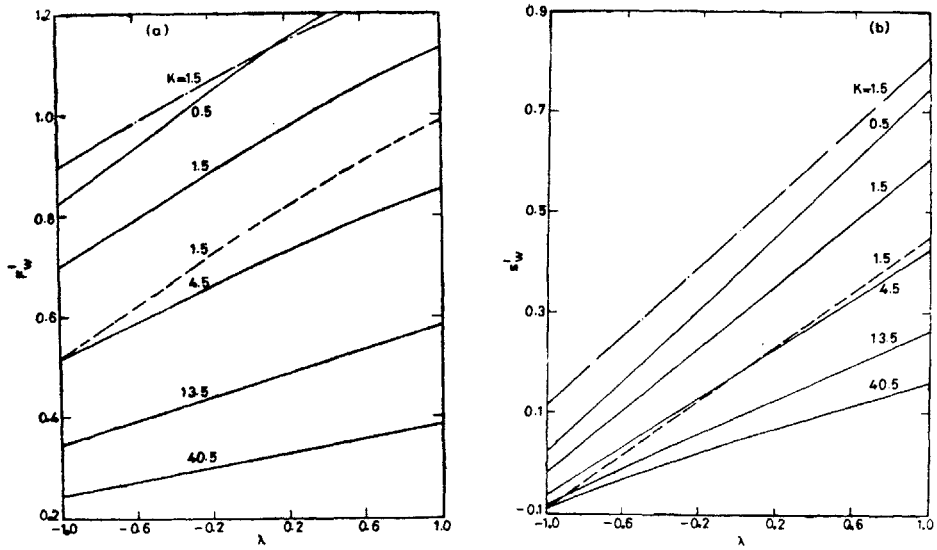
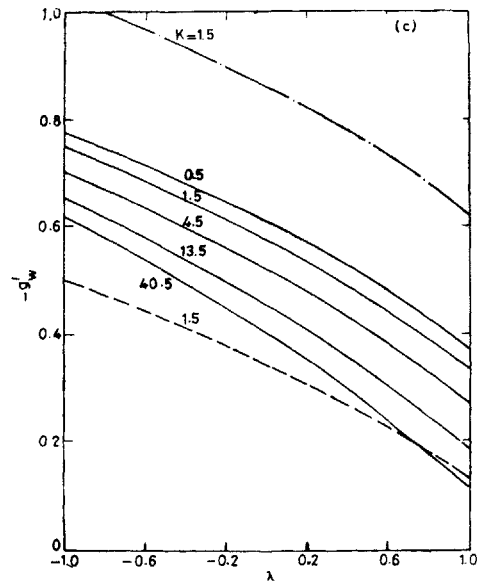


FIG. 1. (a) Skin-friction parameter F'_w (b) microrotation gradient parameter S'_w



and (c) heat-transfer parameter — g'_w
for $G = \sigma = 1.5$, $Pr = 0.7$, $\alpha = 1$._____, $f_w = 0$;_____._____, $f_w = 0.5$; — — —, $f_w = -0.5$ (two-dimensional flow).

K and surface mass transfer parameter f_w on the skin-friction parameter F'_w microrotation gradient s'_w and heat-transfer parameter $-g'_w$ for the range of values of the unsteady parameter λ ($-1 \leqslant \lambda \leqslant 1$) are given in Figs. 1 (a) – (c) and the effects of the microrotation parameter G in Figs. 2 and 3. The results indicate that the skin-friction parameter F'_w and microrotation gradient parameter s'_w increase with λ, but the heat-transfer parameter $-g'_w$ decreases with λ, whatever may be the values of K, f_w and G. For a given λ, the skin-friction parameter F'_w, microrotation gradient parameter s'_w and the heat-transfer parameter $-g'_w$ decrease with coupling parameter K and microrotation parameter G except the heat-transfer parameter $-g'_w$ which increases with G. Similar behaviour has been observed by investigators[6,8,9]. The reason of this reduction is the thickening of momentum boundary layer, microrotation boundary layer and thermal boundary layer thicknesses as the parameter K or G increases. However, the thermal boundary layer thickness decrease as G increases. It is found that

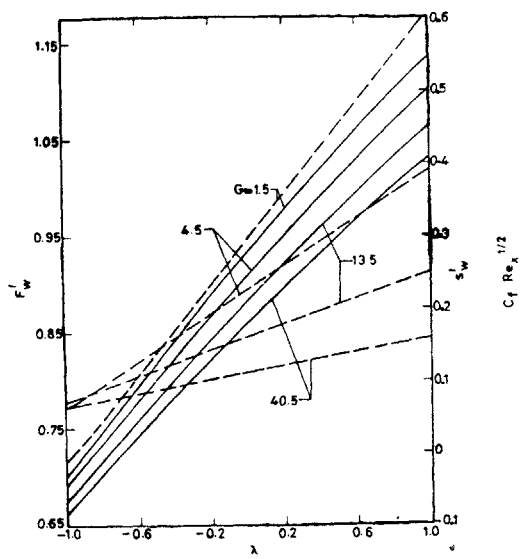


FIG 2. Skin-friction parameter F'_w and microrotation gradient parameter s'_w for $K = \sigma = 1.5$, $f_w = 0$, $\alpha = 1$. ——, F'_w — — — —, s'_w (two-dimensional flow).

the skin-friction, microrotation gradient, and heat-transfer parameters $\left(F_w', s_w', -g_w'\right)$ are strongly dependent on the coupling parameter K, but weakly dependent on microrotation parameter G except the microrotation gradient s_w'. As expected the suction increases the skin-friction, microrotation gradient and heat-transfer parameters $\left(F_w', s_w', -g_w'\right)$, but injection does the reverse. This is because of the reduction in momentum, microrotation and thermal boundary layer thicknesses due to suction which in turn increases F_w', s_w' and $-g_w'$. The effect of injection is just the reverse.

Figure 3 also shows the effect of the Prandtl number Pr on the heat-transfer parameter $-g_w'$. The skin friction and microrotation gradient $\left(F_w', s_w'\right)$ are unaffected by Pr. It is found that the Prandtl number exerts a strong influence on the heat-transfer parameter $-g_w'$ for all values of λ and the heat-transfer $-g_w'$ increases quite rapidly with Pr. The increase in the Prandtl number results in thinner thermal boundary layer thickness which gives rise to large temperature gradient at the wall and hence large heat transfer.

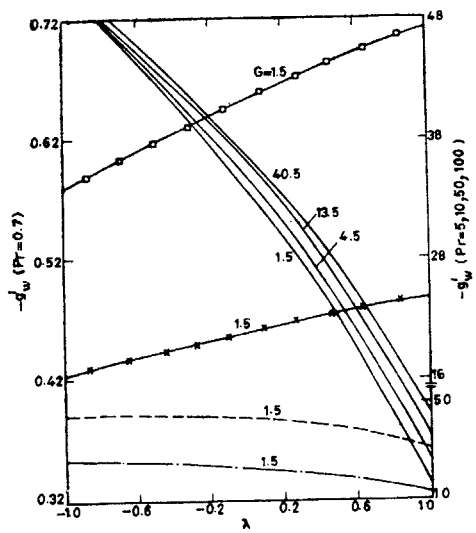


FIG. 3. Heat-transfer parameter $-g_w'$ for $K = \sigma = 1.5$, $f_w = 0$, $\alpha = 1$. _____ $Pr = 0.7$; _____ · _____, $Pr = 5$; — — —, $Pr = 10$; _____ × _____, $Pr = 50$; _____ □ _____, $Pr = 100$ (two-dimensional flow).

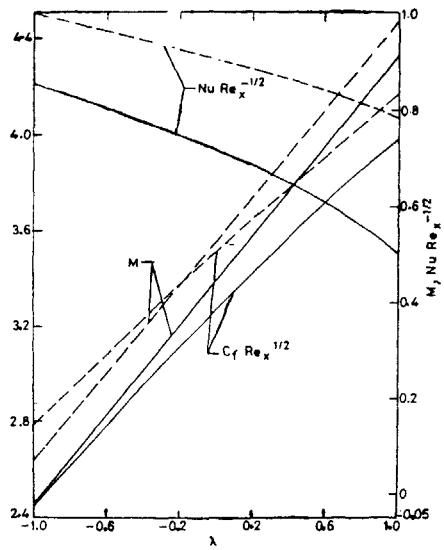


FIG. 4. Skin-friction coefficient $C_f \bar{R}e_x^{1/2}$, Nusselt number $Nu\ \bar{R}e_x^{-1/2}$ and couple stress coefficient M for $K = G = \sigma = 1.5$, $f_w = 0$, $\alpha = 1$, $\beta = 0.3$, $Pr = 0.7$, ____, $j = 0$ (two-dimensional flow); — — —, $j = 1$ (axisymmeteric flow).

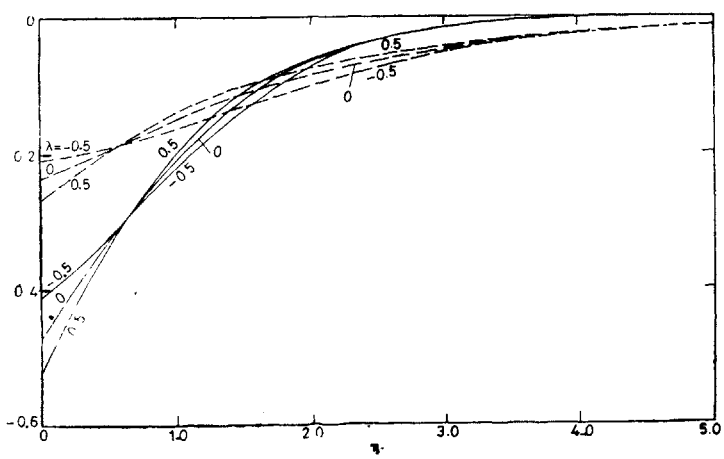
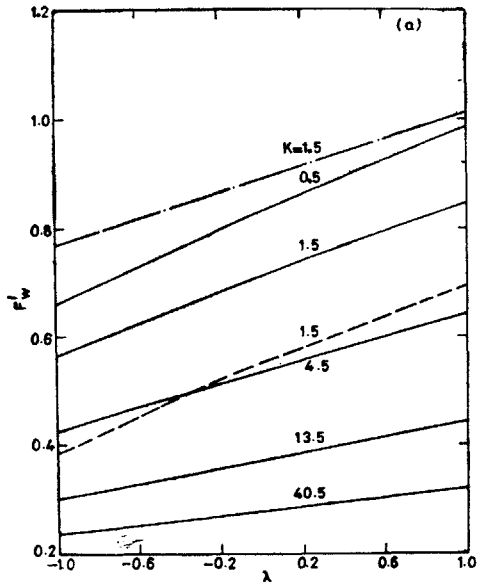


FIG. 5. Microrotation profile s for $\sigma = G = 1.5$, $f_w = 0$, $\alpha = 1$. ____, $K = 1.5$; — — —, $K = 13.5$ (two-dimensional flow).



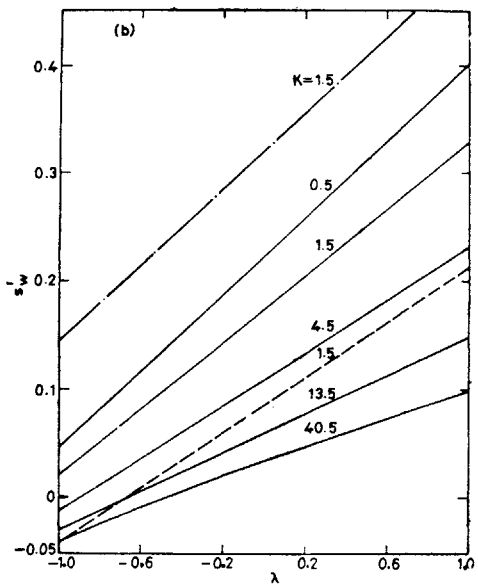
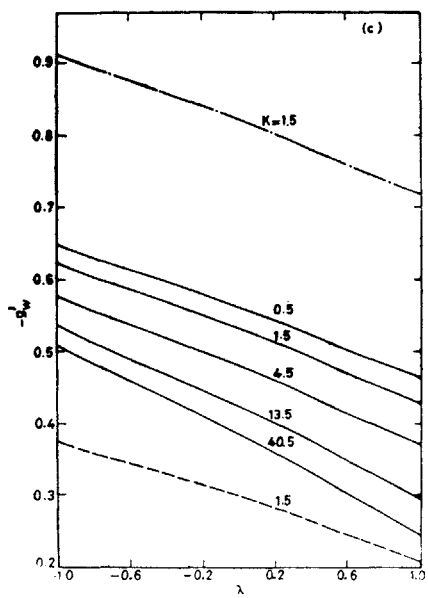


FIG. 6. (a) Skin-friction paramcter — g'_w . (b) microrotation gradient parameter s'_w .



and (c) heat-transfer parameter — g'_w

for $G = \sigma = 1.5$, $Pr = 0.7$, $\alpha = 1$ _____, $f_w = 0$; _____ . _____ , $f_w = 0.5$; — — —, $f_w = -0.5$ (axisymmetric flow).

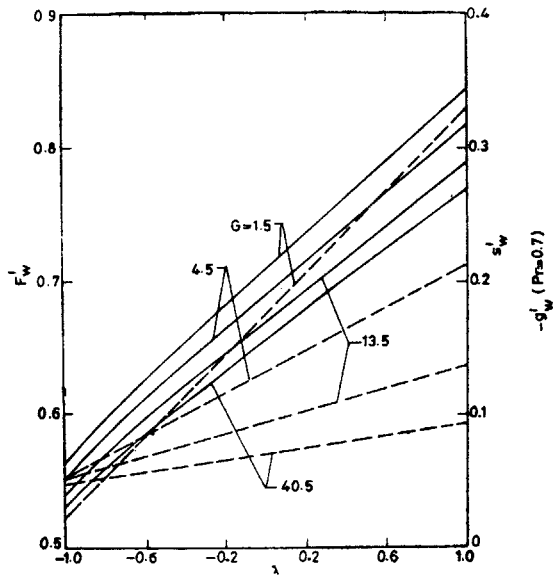


FIG. 7. Skin-friction parameter F'_w and microrotation gradient parameter s'_w for $K = \sigma = 1.5$, $f_w = 0$, $\alpha = 1$. _____, F'_w — — ——, s'_w (axisymmetric flow).

Figure 4 shows the skin-friction coefficient C_f, the couple stress coefficient M and the Nusselt number Nu for both two-dimensional ($j = 0$) and axisymmetric cases ($j = 1$). It is observed that the skin friction and couple stress coefficients (C_f, M) and Nusselt number Nu for the axisymmetric flows are more than those of two-dimensional flows. This is true for all values of the unsteady parameter λ.

The microrotation profiles is shown in Fig. 5. It is observed that the microrotation profiles s at any two values of λ cross each other near the wall (i.e., $\eta < 0.6$) whereas the velocity profile F cross each other towards the edge of the boundary layer (F profile is not shown here to reduce the number of figures). Similar trend has been observed by Yang[15] for the velocity profile for unsteady two-dimensional flow for Newtonian fluids. It may be noted that for Newtonian fluids $s = 0$.

The effects of the parameters K, G, f_w and Pr on the skin friction, gradient of microrotation and heat transfer parameters $\left(F'_w, s'_w, -g'_w\right)$ on the axisymmetric flow ($j = 1$) have been studied and presented in Figs. 6-8. Since the results are qualitatively similar to those of the two-dimensional flow ($j = 0$), they are not discussed here.

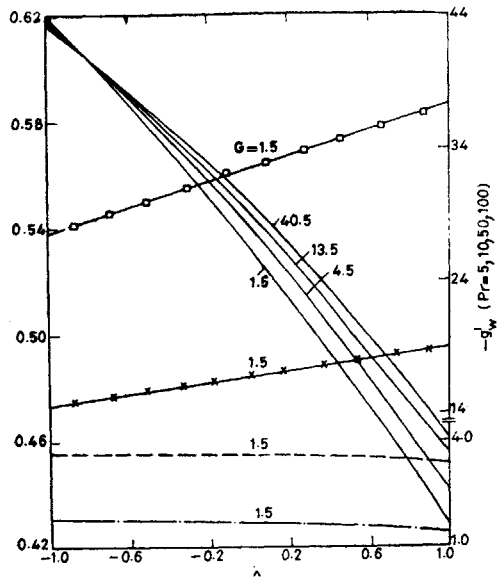


FIG. 8. Heat-transfer parameter $-g_w^*$ for $K = \sigma = 1.5$, $f_w = 0$, $\alpha = 1$. _____, _____, $Pr = 0.7$; _____. _____, $Pr = 5$; — — — —, $Pr = 10$; _____ × _____; $Pr = 50$; _____ □ _____, $Pr = 100$ (axisymmetric flow).

4. CONCLUSIONS

The skin-friction, couple stress and heat transfer coefficients for axisymmetric flows are found to be more than those of two-dimensional flows. The Prandtal number strongly affects the heat transfer, but the skin friction and gradient of microrotation are unaffected by it. The skin friction gradient of microrotation and heat transfer are, in general, change significantly with the unsteady parameter. The coupling parameter K and the mass transfer parameter f_w strongly affect the skin friction, microrotation gradient and heat transfer parameters, but the effect of microrotation parameter on them is comparatively small except for microrotation gradient.

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