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THE REFLECTION OF PLANE GRAVITY WAVES BY UNDULATIONS ON THE SEABED IN PRESENCE OF SURFACE TENSION

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1. INTRODUCTION

The problems of the reflection of incident surface waves by obstacles on the sea bed, such as engineering works, naturally occurring obstacles, such as sand ripples, have been studied by many theoreticians. Lamb[8] obtained the solution for the propagation of surface waves over obstacles considering the case of long waves passing over a finite step, from one constant depth to another. He obtained the expressions for the reflection and transmission coefficient on the basis of continuity of mass and surface elevation but no detailed nature of the flow was obtained. Kreisel[7] obtained a solution of the problem of reflection by obstacles of scale comparable to the surface wavelength. These solutions were restricted generally to simple topographic variations. Lamb's results have been generalized by Bartholomeusz[1], Newman[11], Miles[10]. Kajiura[6] and Hamilton[5] have studied long wave reflection by sea bed topography. Another study was made by Long[9] who studied a more general problem with an arbitrary surface wave propagation over an arbitrary spectrum of bottom perturbation. Davies[2] has considered this type of problem which involves the deterministic treatment of a simpler two-dimensional situation in which reflection and transmission of incident wave energy may occur.

But in none of the above discussions the effect of surface tension on the wave reflection was considered. Recently Evans[4] Rodhes-Robinson[12], and Debnath[3] included the effect of surface tension in the study of water wave problem where some of the works were concerned with the solution involving submerged sources, singularities or obstacles and the localized disturbance at the free surface of water. Here in the present paper we consider the problem of reflection of incident surface waves by undulations on the sea bed in a finite region allowing the effect of surface tension.

Among many of the above problems we consider the problem of Davies[2] under the action of surface tension.

This theory predicts the effect of surface tension on the perturbation potential and on the ratio of the amplitudes of perturbed potential and first order velocity potential. The theory has been applied for a patch of ripples on an otherwise flat bed. Graphs are drawn to study the effect of surface tension on the said ratio. This solution also predicts a resonance effect in back scatter of incident wave as in the theories of Long[9] and Davies[2].

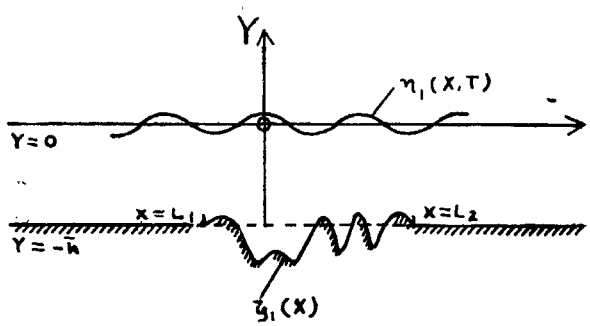
2. The Steady State Formulation and Solution

We consider the unsteady wave motion of an inviscid incompressible, homogeneous liquid with a free surface under the action of gravity in presence of surface tension. Here we consider the reflection of incident surface waves by the undulations on the sea bed in a finite region. The linear perturbation theory has been applied for the present study.

The bed surface is prescribed with its mean level $Y = -\bar{h}$ as

$$\zeta_1(x) = \begin{cases} 0 & -\infty < X \leqslant L_1 \\ Y_-(X) \text{ in } & L_1 \leqslant X \leqslant L_2 \\ 0 & L_2 \leqslant X < \infty. \end{cases}$$

We assume that $\eta_1(X, T)$ represents free-surface elevation, $\phi_1(X, Y, T)$ first order velocity potential, and $\varphi_1(X, Y, T)$ second order perturbation potential.



The linearised equations of motion and the boundary conditions, correct to second order, including the effect of surface tension T' are

$$\varphi_{1XX} + \varphi_{1YY} = 0, \quad -\bar{h} \leqslant Y \leqslant 0, \quad -\infty < X < \infty$$

$$\eta_{1T} + \varphi_{1Y} = 0, \quad Y = 0$$

$$g\eta_1 - \varphi_{1T} - \frac{T'}{\rho}\,\eta_{1XX} = 0,\ Y = 0$$

and

$$-\phi_{1X}\,\zeta_{1X} + \zeta_1\,\phi_{1YY} + \varphi_{1Y} = 0,\ Y = -\bar{h}$$

where g is the gravitational acceleration, ρ the density of liquid and the subscript indicates the differentiation with respect to the variables mentioned.

Introducing the non dimensional variables.

$$(x, y, h) = \frac{1}{L}(x, Y, \bar{h});\ t = \sqrt{\frac{g}{L}}\,T;\ \eta = \frac{1}{L}\,\eta_1$$

$$\phi = \frac{1}{\sqrt{gL^3}}\,\phi_1,\ \varphi = \frac{1}{\sqrt{gL^3}}\,\varphi_1,\ \zeta(x) = \frac{1}{L}\,\zeta_1(x)$$

and the dimensionless parameter.

$T^* = \dfrac{1}{L^2 g}\,\dfrac{T'}{\rho}$, where L is the characteristic length, we rewrite the bed surface topography and the equations of motion as

$$\zeta(x) = \begin{cases} 0 & -\infty < x \leqslant l_1 \\ y_b(x) \text{ in} & l_1 \leqslant x \leqslant L_2 \\ 0 & l_2 \leqslant x < \infty \end{cases} \qquad \ldots(2.1)$$

$$\varphi_{xx} + \varphi_{yy} = 0,\ -h \leqslant y \leqslant 0,\ -\infty < x < \infty \qquad \ldots(2.2)$$

$$\eta_t + \varphi_y = 0,\quad y = 0 \qquad \ldots(2.3)$$

$$\eta - \varphi_t - T^*\,\eta_{xx} = 0,\ y = 0 \qquad \ldots(2.4)$$

$$\varphi_y + \zeta\phi_{yy} - \phi_x\,\zeta_x = 0,\quad y = -h. \qquad \ldots(2.5)$$

Using (2.1) we rewrite (2.5) in the form

$$\varphi_y = \begin{cases} 0 & -\infty < x \leqslant l_1 \\ -v(x, t) \text{ in} & l_1 \leqslant x \leqslant l_2 \\ 0 & l_2 \leqslant x < \infty \end{cases} \text{ on } y = -h$$

where

$$V(x, t) = -\phi_x(x, -h, t)\,\zeta_x + \phi_{yy}(x, -h, t)\,\zeta. \qquad \ldots(2.6)$$

Equations (2.2)−(2.4) give the effect of this disturbance on the fluid modified by the surface tension. We shall consider the asymptotic behaviour of φ as $| x | \to \infty$ and for this we require a solution corresponding to an outgoing wave in this limit.

The steady state solution of eqns. (2.2)−(2.5) becomes indeterminate when we assume the first order velocity potential to be a periodic function of time and so we solve the problem following Lamb[8]. Applying the Fourier transformation with respect to x defined by

$$\tilde{\phi}(k, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y, t)\, e^{ikx}\, dx$$

we transform (2.2)−(2.5) and obtain

$$\tilde{\phi}_{yy} - k^2 \tilde{\varphi} = 0, \quad -h \leqslant y \leqslant 0, \; -\infty < k < \infty$$

$$\tilde{\eta}_t + \tilde{\varphi}_y = 0, \quad y = 0$$

$$\tilde{\eta} - \tilde{\varphi}_t + T^* k^2 \tilde{\eta} = 0, \quad y = 0$$

and

$$\tilde{\phi}_y + \tilde{\zeta}\, \tilde{\phi}_{yy} + k^2\, \tilde{\phi}\, \tilde{\zeta} = 0, \quad y = -h.$$

Eliminating $\tilde{\eta}$, we get

$$\tilde{\phi}_{tt} + (1 + T^* k^2)\, \tilde{\phi}_y = 0, \; y = 0.$$

Following Lamb we introduce a small amount of friction into the problem and find the convergence of the integrals and the uniqueness of the solution though the coefficient of friction is set ultimately to zero. We take the friction as $\mu\varphi_t$ which on Fourier Transformation becomes $\tilde{\mu\varphi}_t$ and the transformed modified surface boundary condition becomes

$$\tilde{\varphi}_{tt} + (1 + T^* k)\, \tilde{\varphi}_y + \mu\, \tilde{\varphi}_t = 0, \quad y = 0 \qquad \ldots(2.7)$$

where μ (> 0) is the coefficient friction.

Also

$$\tilde{\varphi}_y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_y (x, -h, t)\, e^{ikx}\, dx = -\frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} V(x, t)\, e^{ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \wedge (k, t), \text{ (say) } y = -h. \qquad \ldots(2.8)$$

From the transformed equation (2.2), we get

$$\tilde{\varphi}(k, y, t) = A(k, t) \cosh(ky) + B(k, t) \sinh(ky). \qquad \ldots(2.9)$$

Since this satisfies (2.7) and (2.8), so

$$A_{tt}(k, t) + (1 + T^* k^2)\, k\, B(k, t) + \mu A_t(k, t) = 0 \qquad \ldots(2.10)$$

$$+ kA(k, t) \sinh(kh) + kB(k, t) \cosh(kh) = \frac{1}{\sqrt{2\pi}} \wedge (k, t). \qquad \ldots(2.11)$$

For the solution let us assume

$$A(k, t), = A_1(k) \cos(\sigma t) + A_2(k) \sin(\sigma t)$$

$$B(k, t) = B_1(k) \cos(\sigma t) + B_2(k) \sin(\sigma t)$$

$$\wedge (k, t) = \wedge_1 (k) \cos(\sigma t) + \wedge_1 (k) \sin(\sigma t)$$

where σ represents the wave frequency. With the help of these expressions we obtain $\varphi(x, y, t)$ from eqn. (2.9) in the form

$$\varphi(x, y, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \Big[\{A_1 \cosh(ky) + B_1 \sinh(ky)\} \cos(\sigma t)$$

$$+ \{A_2 \cosh(ky) + B_2 \sinh(ky)\} \sin(\sigma t) \Big] e^{-ixk} dk. \qquad \ldots(2.12)$$

As $t > 0$, we have

$$A_1 = \frac{\tilde{a} \wedge_1 + \tilde{b} \wedge_2}{\sqrt{2\pi}\,(\tilde{a}^2 + \tilde{b}^2)}$$

$$A_2 = \frac{\tilde{a} \wedge_2 - \tilde{b} \wedge_1}{\sqrt{2\pi}\,(\tilde{a}^2 + \tilde{b}^2)}$$

$$B_1 = \frac{\sigma^2 A_1 - \mu \sigma A_2}{k (1 + T^* k^2)}$$

$$B_2 = \frac{\sigma^2 A_2 + \mu \sigma A_1}{k (1 + T^* k^2)}$$

in which

$$\tilde{a}(k) = \frac{\sigma^2 \cosh (kh)}{1 + T^* k^2} - K \sinh (kh)$$

$$\tilde{b}(k) = \frac{\mu \sigma \cosh (kh)}{1 + T^* k^2}.$$

3. Evaluation of $\varphi (x, y, t)$

The Fourier inversion in (2.12) can be carried out by means of contour integration combined with the theory of residues. Considering the effect of linear friction term on the singularities in (2.12) we take the contour as described by Davies[2] and $\varphi (x, y, t)$ is evaluated only in the two asymptotic limits $x \to \pm \infty$. The simple poles in (2.12) are given by $\tilde{a}^2 + \tilde{b}^2 = 0$ and in the $s = k + i\gamma$ plane the poles ($s = s_p$) corresponding to $\tilde{a}^2 + \tilde{b}^2 = 0$ lie near the poles ($s = s_0$) given by $\tilde{a}^2 = 0$ since $\tilde{b}(k) \sim \mu$ where μ is small. Therefore, we have

$$s_0 \left(1 + T^* s_0^2\right) \tanh (s_0 h) = \sigma^2. \qquad \ldots(3.1)$$

Also, for the wave number k_1, the dispersion relation for the surface waves is introduced as

$$k_1 \left(1 + T^* k_1^2\right) \tanh (k_1 h) = \sigma^2. \qquad \ldots(3.2)$$

Then (3.1) is satisfied by $s_0 = k_0 = \pm k_1$ on the real axis of s. There are infinite number of poles $s_0 = i\gamma_0$ on the imaginary axis of the complex s-plane where

$$-\gamma_0 \left(1 - T^* \gamma_0^2\right) \tan (\gamma_0 h) = \sigma^2.$$

Now writing $s_p = s_0 + \alpha\mu$, we can determine α by equatnig the coefficients of μ in

$$\sigma^2 \cosh (s_p h) - s_p \left(1 + T^* s_p^2\right) \sinh (s_p h) = \pm i\mu\sigma \cosh (s_p h). \ldots (3.3)$$

Thus

$$\alpha = \pm \frac{i\sigma s_0 \left(1 + T^* s_0^2\right)}{\sigma^4 h - \left(3 s_0^2 T^* + 1\right)\sigma^2 - s_0^2\left(1 + T^* s_0^2\right)h}$$

and

$$s_p = s_0 \left[1 \pm \frac{i\mu\sigma \left(1 + T^* s_0^2\right)}{\sigma^4 h - \left(3 s_0^2 T^* + 1\right)\sigma^2 - s_0^2\left(1 + T^* s_0^2\right)h} \right]$$

We obtain two distinct poles lying slightly above and below the k-axis for each original pole on the real axis of s due to the effect of friction. Similar effect on the poles lying on the imaginary axis of s are also observed. For the convergence of the solution we take the contour in the lower or upper half plane according as $x \to +\infty$ or $x \to -\infty$. To get the original path of integration $(-\infty, \infty)$, the radius of the semicircular portion of each contour is assumed to tend to infinity. The residues R_{s_p} at a simple pole $s = s_p$ is give by

$$R_{s_p} = \underset{s \to s_p}{\alpha t} \left[(s - s_p)\, X \frac{1}{\sqrt{2\pi}} \left[\{A_1 \cosh(sy) + B_1 \sinh(sy)\} \cos(\sigma t) \right.\right.$$

$$\left.\left. + \{A_2 \cosh(sy) + B_2 \sinh(sy)\} \sin(\sigma t) \right] e^{-isx} \right]$$

$$= -\frac{1}{2\pi} R_* \Big|_{s_p} \left[\{A_{1*} \cosh(sy) + B_{1*} \sinh(sy)\} \Big|_{s_p} \cos(\sigma t) \right.$$

$$\left. + \{A_{2*} \cosh(sy) + B_{2*} \sinh(sy)\} \Big|_{s_p} \sin(\sigma t) \right] e^{-is_p x}$$

where

$$D_* = \frac{\sqrt{2\pi}\,(\tilde{a}^2 + \tilde{b}^2)}{\tilde{a}} D,\ D \text{ may be } A_1, A_2, B_1 \text{ or } B_2;$$

and

$$R_* \Big|_{s_p} = \underset{s \to s_p}{\alpha t} \frac{-\tilde{a}(s)(s - s_p)}{\tilde{a}^2(s) + \tilde{b}^2(s)}$$

(*equation continued on p. 252*)

$$= \frac{\left(1 + T^* S_p^2\right) \cosh (S_p h)}{\left(1 + T^* S_p^2\right) 2 s_p h + \left(1 + 3\ T^*\ s_p^2\right) \sinh (2 S_p h)}$$

In the asymptotic limit $x \to -\infty$, the contour contains the poles lying only in the upper half plane and the residue contribution gives the velocity potential as

$$\varphi (x, y, t) = 2\pi i \left(R_{s_{p1}} + R_{s_{p2}}\right) \mid_{\mu=0}$$

where

$$s_{p1} = K_1 + i\mu K_2 \text{ and } s_{p2} = -K_1 + i\mu K_2 \ (k_2 > 0).$$

The residue contribution from the poles close to the imaginary axis of the complex s-plane leads to the non propagating modes which decay asymptotically in x as $\mu \to 0$ and so has not been considered here and we have been concerned only with the propagating modes with the modes with the limit $\mu \to 0$.

Finally, as $x \to -\infty$, we have the following representation for the velocity potential

$$\varphi (x,y, t) = i\, \Omega (y)\, [(\wedge_1 - i \wedge_2) \mid_{-k_1} e^{i(k_1 x + \sigma t)} - (\wedge_1 + i\ \wedge_2) \mid_{k_1} e^{i(k_1 x + \sigma t)}] \qquad \ldots(3.4)$$

where

$$\Omega (y) = \frac{\left(1 + T^*\ K_1^2\right) \cosh K_1 (y + h)}{\left(1 + T^*\ k_1^2\right) 2k_1 h + \left(1 + 3\ T^*\ k_1^2\right) \sinh (2k_1 h)}\ .$$

In the asymptotic limit $x \to \infty$ when the poles lie in the lower half plane we represent similarly the velocity potential as

$$\varphi (x, y, t) = -i\, \Omega (y)\, [(\wedge_1 + i\ \wedge_2) \mid_{-k_1} e^{i(k_1 x - \sigma t)} - (\wedge_1 - i \wedge_2) \mid_{k_1} e^{-i(k_1 x - \sigma t)}]. \qquad \ldots(3.5)$$

4. Application of the Theory

We discuss the application of the theory in the case when progressing surface waves are incident upon sinusoidal bed forms of finite horizontal extent. The problem was discussed by Davies[2] neglecting the effect of surface tension. We specify ϕ first in the absence of any bed form and then consider the interaction of $\phi\ (x, y, t)$ with the patch of ripples on the bed influenced by the surface tension. We assume

$$\phi(x, y, t) = \frac{a}{\sigma} \frac{\cosh k_1 (y + h)}{\cosh k_1 h} \cos (K_1 x - \sigma t) \qquad \ldots(4.1)$$

and the bed topography as

$$y_b(x) = b \sin (lx + \delta) \text{ in } l_1 \leqslant x \leqslant l_2. \qquad \ldots(4.2)$$

Where the non-dimensional quantities a, b, σ, h and δ represent the amplitude, of surface waves, ripple amplitude, frequency of surface waves, depth of water and the constant phase angle respectively. Also for the region of bed forms we take

$$l_1 = (-h\pi - \delta)/l \text{ and } l_2 = (m\pi - \delta)/l$$

where we consider $\frac{n + m}{2}$ ripples of wave number l in the patch.

Then

$$V(x, t) = -\phi_y \mid_{-h}$$

$$= Q_* [\{l \sin (k_1 x) \cos (lx+\delta) + k_1 \cos (k_1 x) \sin (lx+\delta)\} \cos (\sigma t)$$

$$+ \{-l \cos (k_1 x) \cos (lx + \delta) + k_1 \sin (k_1 x) \sin (lx + \delta)\} \sin (\sigma t)]$$

where

$$Q_* = \frac{ab\, k_1}{\sigma \cosh (k_1 h)}.$$

Hence for the case we obtain the perturbation potential $\varphi(x, y, t)$ in the two asymptotic limit $x \to \pm \infty$ in the form

$$\varphi(x, y, t) = -Q_* \Omega(y) \frac{2k_1/l}{\left(\frac{2k_1}{l}\right)^2_{-1}} \left[(-1)^m \sin (k_1 x + \sigma t - 2k_1 l_2) - (-1)^n \sin (k_1 x + \sigma t - 2k_1 l_1)\right]$$

$$x \to -\infty \qquad \ldots(4.3)$$

$$\varphi(x, y, t) = Q_* \Omega(y) \frac{2k_1}{l} \left[(-1)^m - (-1)^n\right] \sin (k_1 x - \sigma t)$$

$$\text{as } x \to +\infty \qquad \ldots(4.4)$$

in which $\Omega(y)$ singnificantly depends on the surface tension, hence $\varphi(x, y, t)$ is affected. Particularly for $m = n$ and $\delta = 0$, we modify the perturbation potential

$$\varphi(x, y, t) = \frac{2\,ab\,k_1}{\sigma \cosh(k_1 h)} \frac{\left(1 + T^* k_1^2\right) \cosh k_1 (y + h)}{\left(1 + T^* k_1^2\right) 2k_1 h + \left(1 + 3\,T^* k_1^2\right) \sinh(2k_1 h)}$$

$$\times \frac{2k_1 l}{\left(\frac{2k_1}{l}\right)^2 - 1} (-1)^m \sin\left(\frac{2k_1}{l} m\pi\right) \cos(k_1 x + \sigma t),$$

$$\text{as } x \to -\infty \qquad \ldots(4.5)$$

$$\varphi(x, y, t)\ 0, \text{ as } x \to +\infty \qquad \ldots(4.6)$$

i.e. the upwave side is only disturbed. Hence in the presence of surface tension,

$$\frac{\text{Amplitude } \varphi(x, y, t) \mid_{x \to -\infty}}{\text{Amplitude } \phi(x, y, t)}$$

$$= \frac{2\,b\,k_1 \left(1 + T^* k_1^2\right)}{\left(1 + T^* k_1^2\right) 2k_1 h + \left(1 + 3\,T^*\ k_1^2\right) \sinh(2k_1 h)}$$

$$\left[(-1)^m \frac{2k_1}{l} \frac{\sin\left(\frac{2k_1}{l} m\pi\right)}{\left(\frac{2k_1}{l}\right)^2 - 1}\right]. \qquad \ldots(4.7)$$

We note from the term in the third bracket in eqn. (4.7) that the amplitude of the reflected wave has an oscillatory nature which oscillates in the ratio of the overall length of the ripple patch to the surface wavelength. The amplitude of the reflected wave increases with the number of ripples and for large number of ripples infinite resonance occurs if we take the bed wave length as half the surface wavelengh i. e. $\frac{2k_1}{l} = 1$. However, this portion of the ratio is not affected by the surface tension. It is interesting to note from eqns. (4.5) that the amplitude of the reflected wave significantly depends on the ripple amplitude, water depth, number of ripples and as well as on the surface tension. Now we are interested to study the effect of surface tension on the reflected waves. For long waves with the wave number satisfying $\frac{2k_1}{l} = 1$ we have

$$\frac{R}{a} = \frac{\text{Amplitude } \varphi(x, y, t) \mid_{x \to -\infty}}{\text{Amplitude } \phi(x, y, t)}$$

(equation continued on p. 255)

$$= \frac{b}{2h} \frac{1 + T^* \frac{l^2}{4}}{1 + T^* \frac{l^2}{2}} \frac{m\pi}{2} \; ; \left(\frac{2k_1}{l} = 1, \; k_1 h \ll 1 \right)$$

which is less the ratio obtained by Davies[2] in absence of surface tension. This shows that the amplitude of the reflected wave decreases in presence of surface tension. Figures 1-4 are drawn in which the effect of the surface tension has been studied with the different values of m and T^* where ripple steepness $bl = \frac{\pi}{10}$. The range of $\frac{b}{h}$

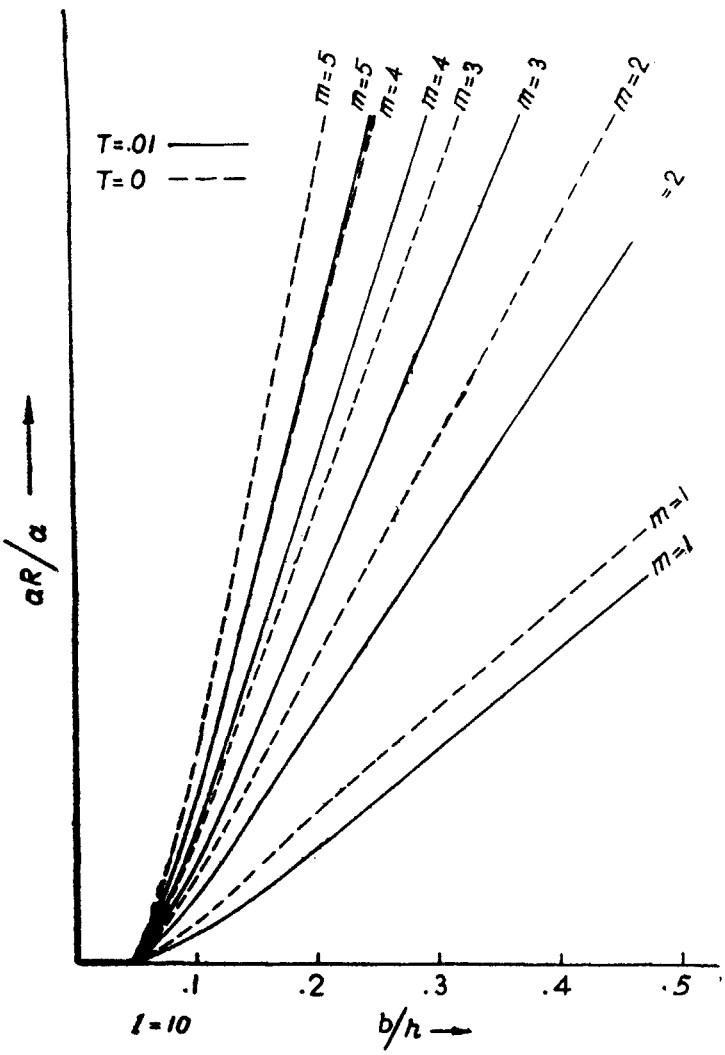


FIG. 1.

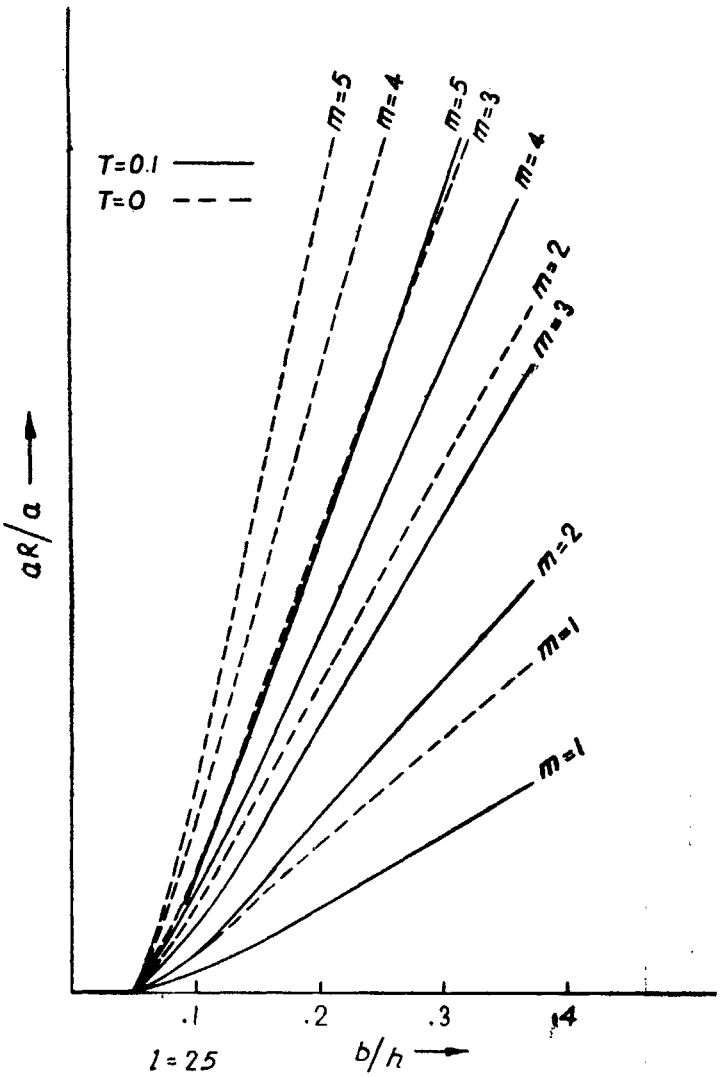


FIG. 2.

is taken from 0 to 0.4 which occurs for natural features. Dotted lines and continuous lines in each of the curve give a comparative study of the effect of surface tension on the reflected wave amplitude. In each of the figures the ratio $\frac{a_R}{a}$ has been plotted for $m = 1$ to 5. In Figs. 1-3 where $T^* = .01$, $\frac{2k_1}{l} = 1$, $bl = \frac{\pi}{10}$, $l = 10,\ 25,\ 100$ cm respectively, we note that large values of l give more deflection of the curves than those of Davies[2] in which surface tension was not considered. Comparing

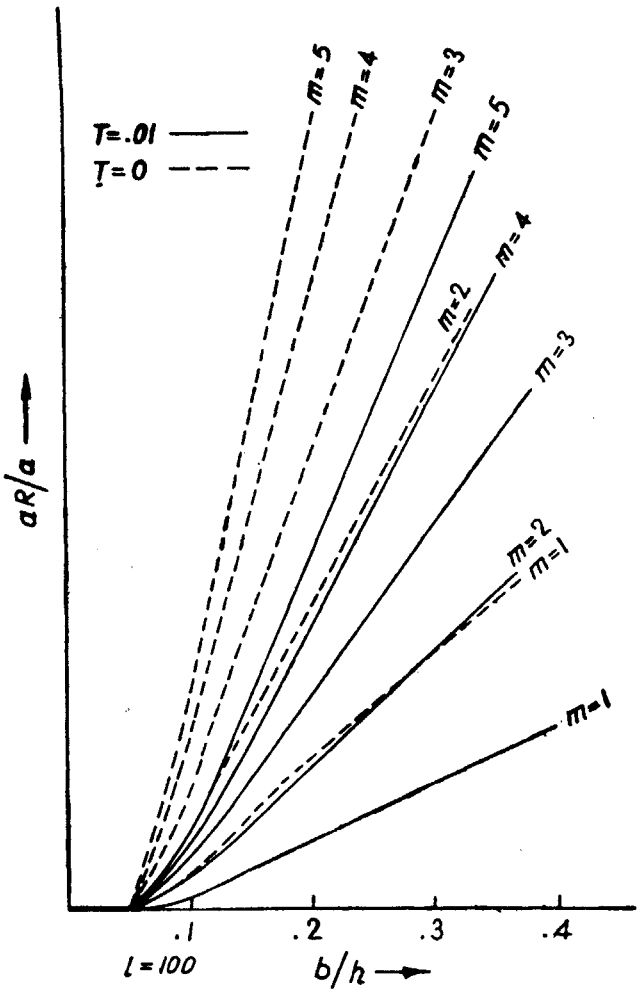


FIG. 3.

Fig. 1 and 4 we note that the deflection of the curve becomes larger with the larger values of T^*. Moreover, the deflection increases with increasing m.

Each curve has zero slope at the origin and for small $\frac{b}{h}$ the slope increases and the curves have the straight line form approximately after $\frac{b}{h} = 0.15$.

Therefore, this graphical study helps us to compare the effect of b, l, h and m on the reflected wave amplitude and the coupling of T^* with b, l, h are also observed by the comparison of Figures 1-4 etc.

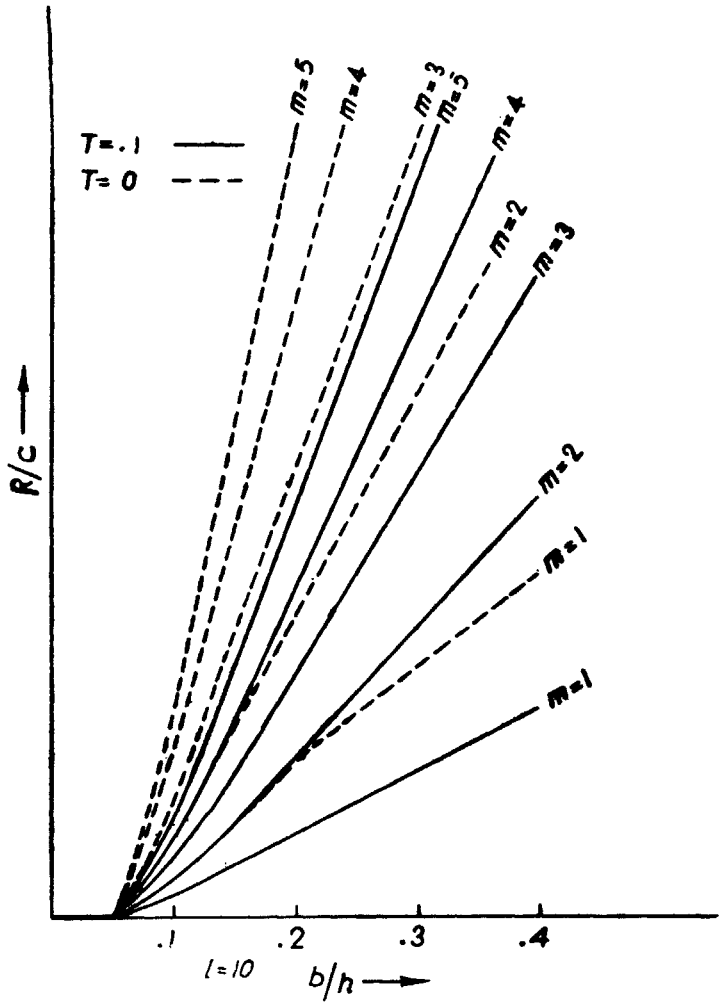


FIG. 4.

CONCLUSION

The results obtained are valid over physically interesting surface waves allowing the effect of surface tension and the bed forms parameters though the analysis has been restricted to two-dimensional irrotational flow. A steady state analysis has been applied to obtain the interaction of the first order motion with the bed forms and the general results (3.4) and (3.5) relate the sizes of outgoing waves to the Fourier transform of the perturbating vertical velocity in presence of surface tension. The reflection coefficient has been found out which is characterized by two properties.

It is oscillatory in the ratio of surface wave length to the overall length at the region of bed forms and it has a pronounced but finite resonance when the surface wave length is approximately twice the bed wave length.

In this critical situation the amplitude of the reflected wave is proportional to the to the number of bed forms in the patch, amplitude of ripples, and surface tension. So the effect of surface tension must be considered for the study of reflection of waves by undulations on sea bed.

The different results reduce to the expressions given by Davies[2] when we put $T^* = 0$.

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