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UNIFORM LAW OF LARGE NUMBERS

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The main object of the paper is to introduce a new concept called uniform law of large numbers, and generalise some known results due to Rohatgi on convergence problem of matrix sums of independent random variables.

§ 1. Given a sequence $(X_k)_{k=1}^{\infty}$ of random variables with $S_n = X_1 + X_2 + \ldots\ldots + X_n$, as partial sums, the classical problem of weak (strong) law of large numbers deals with weighted $(C, 1)$ mean :

$$\frac{S_n}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n} . \qquad \ldots(1)$$

Since $(C, 1)$ mean is translative, it is immaterial if we consider the convergence of

$$\frac{S_{n+j-1} - S_{j-1}}{n} = \frac{X_j + X_{j+1} \ldots + X_{n+j-1}}{n} \qquad \ldots(2)$$

for any 'fixed' j.

In a random experiment the ultimate result is considered independent of the time of observation by an individual. Hence it is perhaps more natural to consider the convergence of the mean in (2) 'uniformly in j, instead of any fixed j.'

This leads to the following concept of the Uniform law of large numbers.

Definition—Uniform weak (strong) law of large numbers is said to hold if,

$$\frac{1}{n}\left(\sum_{k=j}^{n+j-1} X_k - \sum_{k=1}^{n+j-1} E(X_k)\right) \to 0 \text{ in Pr. (a.s).}$$

Generalising the law of large numbers Pruitt[6] and Rohatgi[7] have considered the convergence in Pr and a.s of the weighted mean

$$\sum_{k=1}^{\infty} a_{nk} X_k \qquad \ldots(3)$$

with suitable regularity condition on the matrix $A = (a_{nk})$. See also Azuma[1], Borwein[2], and Stout[8], where weighted sums of the form (3) have been considered.

Lorentz[5] has shown that the convergence of the mean in (2) when it is uniform with respect to j, cannot be obtained as a particular case of the mean (3). But by considering a sequence of matrices $A^j = \left(a_{nk}^j\right)$ as in Bell[2], it is possible to generalise both the mean given in (1) and (2) by restricting respectively to the following matrices:

$$a_{nk} = \begin{cases} \frac{1}{n} & (1 \leqslant k \leqslant n), \\ 0 & (n > k) \quad ; \end{cases}$$

$$a_{nk}^j = \begin{cases} \frac{1}{n} & (j \leqslant k \leqslant j + n - 1), \\ 0 & \text{(elsewhere)}. \end{cases}$$

Now generalising both the concepts of the law of large numbers and uniform law of large numbers as introduced above, we consider the weighted sums of three parameter matrices given by $A^j = \left(a_{nk}^{(j)}\right)$ and define the sum

$$Y_n^j = \sum_{k=1}^{\infty} a_{nk}^j X_k \qquad \ldots(4)$$

provided the series on the right converges almost surely for each n and j.

Definition—If there exists a constant μ such that

$$Y_n^j = \sum_{k=1}^{\infty} a_{nk}^j X_k \to \mu \text{ (a.s)}$$

as $n \to \infty$, uniformly in j, then we say that the weighted sum of X_k's obeys the 'uniform strong law of large numbers'. If for any $\delta > 0$

$$\lim_n \sup_j P\left(\left|Y_n^j - \mu\right| > \delta\right) = 0$$

then we say that the weighted sum of (X_k) obeys the uniform weak law of large numbers. In that case we write

$$Y_n^j \to \mu \text{ (u. Pr)} .$$

§2. We now prove two theorems concerning uniform weak law of large numbers. Theorem 1 deals with independent identically distributed random variables while Theorem 2 deals with only independent random variables.

Theorem 1—Let $(X_k)_{k=1}^{\infty}$ be a given fixed sequence of independent identically distributed non-degenerate random variables having expectation each equal to μ, such that $E \mid X_k \mid < \infty$.

Let $A = \left(a_{nk}^j\right)$ be a three parameter matrix such that

$$\|A\| = \sup_{n,j} \sum_{k=1}^{\infty} \left|a_{nk}^j\right| < \infty \qquad \ldots(5)$$

and there exists α for which, uniformly in j,

$$\sum_{k=1}^{\infty} a_{nk}^j \to \alpha \text{ (as } n \to \infty). \qquad \ldots(6)$$

Then in order that the sequence of numbers (α_k) should exists with

$$B = \sum_{k=1}^{\infty} \mid \alpha_k \mid < \infty$$

such that

$$\sum_{k=1}^{\infty} \left(a_{nk}^j - \alpha_k\right) X_k \to D \text{ (u. Pr)}$$

as $n \to \infty$ for some constant D, it is necessary and sufficient that

$$\lim_{n} \sup_{k,j} \left|a_{nk}^j - \alpha_k\right| = 0. \qquad \ldots(7)$$

If (7) holds then D is given by

$$D = \mu\,(\alpha - \sum_{k=1}^{\infty} \alpha_k).$$

Remark: At first sight, the theorem looks as if it is more or less analogous to the standard Teoplitz theorem. In fact, as far as the necessity part is concerned, there is a fundamental difference. In the Teoplitz theorem, the conditions are necessary in order that the conclusion should hold for every convergent sequence (and this is, of course, essential for the proof). Here we take a fixed X_k, and assume only that the conclusion holds for that particular X_k. Further the necessity of (7) has been proved assuming (6). We are thankful to the referee to raise the question if it is possible to prove that (6) is necessary. However, we have not yet been able to show this. Before we state the next theorem, we state a definition.

The sequence of random variables (X_k) is said to be uniformly bounded by a random variable X if

$$P(|X_k| > x) \leqslant P(|X| \geqslant x) \text{ for all } x > 0.$$

If $E|X|^r < \infty$ for some $0 < r < \infty$, it follows by a simple rearrangement of the series that $E|X_k|^r \leqslant E|X|^r < \infty$ for $k = 1, 2, 3, \ldots$.

Theorem 2 – Let $0 < r \leqslant 1$. Suppose that (X_k) be a sequence of independent random variables uniformly bounded by the random variable X such that $E|X|^r < \infty$ and $EX_k = 0$ for all k. Suppose the matrix $A = \left(a_{nk}^j\right)$ satisfies (7) and such that

$$\sup_{n,j} \sum_{k=1}^{\infty} \left|a_{nk}^j\right|^r < \infty. \qquad \ldots(8)$$

Then

$$\sum_{k=1}^{\infty} \left(a_{nk}^j - \alpha_k\right) X_k \to 0 \text{ (u. Pr)}.$$

§3. *Proof of Theorem* 1 (*Sufficiency*)—Since

$$E \sum_{k=1}^{\infty} \left|\left(a_{nk}^j - \alpha_k\right) X_k\right| < (\|A\| + B)\, E|X_k| < \infty$$

the series

$$Y_n^j = \sum_{k=1}^{\infty} \left(a_{nk}^j - \alpha_k \right) X_k$$

exists almost surely for each n and j.

Define

$$X_{nk}^j = \begin{cases} \left(a_{nk}^j - \alpha_k \right) X_k, \text{ if } \left| \left(a_{nk}^j - \alpha_k \right) X_k \right| < 1 , \\ 0 \text{ , if } \left| \left(a_{nk}^j - \alpha_k \right) X_k \right| \geqslant 1 . \end{cases}$$

Put $Z_n^j = \sum_{k=1}^{\infty} X_{nk}^j$. Let X be the random variable such that F is the common distribution of X and X_k's. Since $E \mid X \mid = E \mid X_k \mid < \infty$ for each k, we have

$$T\, P (\mid X \mid \geqslant T) \to 0 \text{ as } T \to \infty.$$

Hence by hypothesis (7), given $\epsilon > 0$, there exists a positive interger n_0 such that for $n > n_0$,

$$P \left(\left| \left(a_{nk}^j - \alpha_k \right) X \right| > 1 \right) \leqslant \epsilon \left| a_{nk}^j - \alpha_k \right| . \qquad \ldots(9)$$

Now by (7) and (9)

$$P \left(Y_n^j . \neq Z_n^j \right) \leqslant \sum_{k=1}^{\infty} P \left(\left(a_{nk}^j - \alpha_k \right) X_k \neq X_{nk}^j \right)$$

$$= \sum_{k=1}^{\infty} P \left(\left| \left(a_{nk}^j - \alpha_k \right) \right| \mid X_k \mid > 1 \right)$$

$$\leqslant \epsilon \sum_{k=1}^{\infty} \left| a_{nk}^j - \alpha_k \right|$$

(equation continued on p. 31)

$$\leqslant \epsilon \left(\sum_{k=1}^{\infty} \left| a_{nk}^{j} \right| + \sum_{k=1}^{\infty} |\alpha_k| \right)$$

$$\leqslant \epsilon (\|A\| + B)$$

for $n > n_0$. Thus Y_n^j and Z_n^j are equivalent sequences. Therefore it is enough to show that

$$Z_n^j \to D = \mu \left(\alpha - \sum_{k=1}^{\infty} \alpha_k \right), \text{ (u. Pr).}$$

Hence it is enough to show that for any $\delta > 0$

$$\sup_j P \left(\left| Z_n^j - \mu \left(\alpha - \sum_{k=1}^{\infty} \alpha_k \right) \right| > \delta \right) \to 0 \text{ (as } n \to \infty).$$

But it can be verified easily that

$$EZ_n^j - \mu \left(\alpha - \sum_{k=1}^{\infty} \alpha_k \right)$$

$$= \sum_{k=1}^{\infty} a_{nk}^{j} \left[\int_{\theta<1} x dF - \mu \right] + \mu \left[\sum_{k=1}^{\infty} a_{nk}^{j} - \alpha \right]$$

$$- \sum_{k=1}^{\infty} \alpha_k \left[\int_{\theta<1} x dF - \mu \right] \qquad \ldots(10)$$

where we write

$$\theta = \theta \ (n, k, j) = \left| a_{nk}^{j} - \alpha_k \right| |X| .$$

Therefore it follows from (10), properties of $A = \left(a_{nk}^{j} \right)$ and hypotheses (6) and (7) that

$$EZ_n^j \to \mu \left(\alpha - \sum_{k=1}^{\infty} \alpha_k \right) \text{ uniformly in } j \text{ as } n \to \infty.$$

Hence there exists a positive integer n_1 such that for $n > n_1 > n_0$ and all j

$$\left| EZ_n^j - \mu\left(\alpha - \sum_{k=1}^{\infty} \alpha_k\right)\right| < \delta/2 .$$

Thus we have

$$P\left(\left|Z_n^j - \mu\left(\alpha - \sum_{k=1}^{\infty} \alpha_k\right)\right| > \delta\right)$$

$$\leqslant P\left(\left|Z_n^j - EZ_n^j\right| + \left|EZ_n^j - \mu(\alpha - \Sigma\, \alpha_k)\right| > \delta\right)$$

$$\leqslant P\left(\left|Z_n^j - EZ_n^j\right| > \delta/2\right) \quad (n > n_1)$$

$$\leqslant \sigma^2\left(Z_n^j\right) \Big/ \frac{\delta^2}{4} \quad \text{(Chebyshev inequality);}$$

where σ^2 denotes the variance. Thus the proof of sufficiency is completed if we show that $\sigma^2\left(Z_n^j\right) \to 0$ as $n \to \infty$ uniformly in j. Note that

$$\frac{1}{T}\int_{|x|<T} X^2\, dF = \frac{1}{T}\left\{ - T^2 P(|X| \geqslant T) + 2\int_0^T xP(|X| \geqslant x)\, dx \right\}. \qquad \ldots(11)$$

Since X_k's are independent

$$\sigma^2\left(Z_n^j \leqslant \sum_{k=1}^{\infty} E\left(X_{nk}^j\right)\right)^2$$

$$\leqslant \sum_{k=1}^{\infty} \left|a_{nk}^j - \alpha_k\right|^2 \int_{\left|a_{nk}^j - \alpha_k\right| |x| < 1} x^2\, dF \qquad \ldots(12)$$

where F is the common distribution of X and X_k's. But as $\sum_{k=1}^{\infty}\left|a_{nk}^j - \alpha_k\right| \leqslant (\|A\| + B)$,

it follows from (11), (12) and hypothesis (7) that $\sigma^2\left(Z_n^j\right) \to 0$ uniformly in j as $n \to \infty$.

This completes the proof.

Necessity—We are given that

$$T_n^j = \sum_{k=1}^{\infty} \left(a_{nk}^j - \alpha_k\right) X_k \to D, \text{ (u. Pr)}$$

for some constant D. Hence it converges in law. We are now to show that

$$\sup_{k,j} \left|a_{nk}^j - \alpha_k\right| \to 0 \text{ as } n \to \infty.$$

By the continuity of e^{iux} as a function of x for fixed u and hypothesis that $T_n^j \to D$ (u.Pr)

$$e^{iuT_n^j} \to e^{iuD} \text{ (u. Pr).}$$

Now if U_n^j is uniformly bounded and $U_n^j \to U$ (u. Pr) then $E\,U_n^j \to U$. As such since $e^{iuT_n^j}$ is uniformly bounded (being of modulus 1), hence if $g(u) = E(e^{iuX_k})$ be the characteristic function of X_k, we obtain

$$\prod_{k=1}^{\infty} g\left(\left(a_{nk}^j - \alpha_k\right)u\right) = E\,e^{iu_k\Sigma\left(\left(a_n^j - \alpha_k\right)X_k\right)}$$

$$= E\,e^{iuT_n^j}$$

$$\to e^{uiD}$$

as $n \to \infty$, uniformly in j.

Note that e^{iuD} is a point on the unit circle. But

$$\left|\prod_{k=1}^{\infty} g\left(\left(a_{nk}^j - \alpha_k\right)u\right)\right| \leqslant \left|g\left(\left(a_{nm}^j - \alpha_m\right)u\right)\right| \leqslant 1$$

for any m for and all j. Therefore for any m,

$$| g \left(\left(a_{nm}^{j} - \alpha_m \right) u \right) | \to 1$$

as $n \to \infty$, uniformly in j and this again implies

$$\inf_m | g \left(\left(a_{nm}^{j} - \alpha_m \right) u \right) | \to 1 \qquad \ldots(13)$$

as $n \to \infty$ uniformly in j. Again since X_k is non-degenerate we can find an u_0 (real) such that given any (sufficient small) $\delta > 0$, there is a constant $a(\delta)$ (say) with $a(\delta) < 1$ such that

$$g(u) < a(\delta) \text{ for } \delta \leqslant | u | < u_0.$$

Let

$$u = \frac{u_0}{2(\|A\| + B)}.$$

Then, for all j and m,

$$\left| \left(a_{nm}^{j} - \alpha_m \right) u \right| \leqslant (\|A\| + B) | u | < \frac{u_0}{2} < u_0$$

so that

$$\sup_{n,j} \left| \left(a_{nm}^{j} - \alpha_m \right) u \right| < u_0.$$

This implies that

$$\left(a_{nm}^{j} - \alpha_m \right) u \to 0$$

uniformly in m and j as $n \to \infty$. For if it were false there would be a $\delta > 0$ such that for some arbitrary large n, there would exist m, j with

$$\left| \left(a_{nm}^{j} - \alpha_m \right) u \right| \geqslant \delta$$

and thus by what just has been said

$$\left| g \left(\left(a_{nm}^{j} - \alpha_m \right) u \right) \right| < a(\delta)$$

in contradiction of (13).

This proofs the necessity of (7).

Since we have now proved that (7) holds, it follows from the sufficiency part that

$$\sum_{k=1}^{\infty} \left(a_{nk}^{j} - \alpha_k \right) X_k \to \mu \left(\alpha - \sum_{k=1}^{\infty} \alpha_k \right) \text{ (u. Pr)}$$

whence the result.

This completes the proof.

Proof of the Theorem 2—Write

$$b_{nk}^{j} = \left(a_{nk}^{j} - \alpha_k \right).$$

Therefore by (7) and (8) and a known inequality (see Loeve[4], p. 155)

$$\left.\begin{aligned} &\sup_{n,j} \sum_{k=1}^{\infty} \left| b_{nk}^{j} \right|^r \leqslant \sup_{n,j} \sum_{k=1}^{\infty} \left(\left| a_{nk}^{j} \right|^r + \left| \alpha_k \right|^r \right) < \infty \\ &\text{and} \\ &\sup_{k,j} \left| b_{nk}^{j} \right| \to 0 \text{ as } n \to \infty. \end{aligned}\right\} \quad \ldots(14)$$

Hence $\sum_{k=1}^{\infty} b_{nk}^{j} X_k$ exists almost surely for each n and j.

Thus we have to prove that

$$P \left(\left| \sum_{k=1}^{\infty} b_{nk}^{j} X_k \right| > \delta \right) \to 0$$

uniformly in j as $n \to \infty$ for any $\delta > 0$.

We define

$$X_{nk}^{j} = \begin{cases} b_{nk}^{j} X_k, \text{ if } \left(\left| b_{nk}^{j} \right| \mid X_k \mid < 1 \right) \\ 0, \text{ if } \left(\left| b_{nk}^{j} \right| \mid X_k \mid \geqslant 1 \right). \end{cases}$$

Put

$$Z_n^j = \sum_{k=1}^{\infty} X_{nk}^j \text{ and } Y_n^j = \sum_{k=1}^{\infty} b_{nk}^j X_k.$$

Since $E \mid X \mid^r < \infty$, we have as $T \to \infty$

$$T^r P(\mid X \mid \geqslant T) \to 0; TP(\mid X \mid \geqslant T^{1/r}) \to 0.$$

Therefore

$$P\left(Y_n^j \neq Z_n^j\right) \leqslant \sum_{k=1}^{\infty} P\left(\left|b_{nk}^j X_k\right| \geqslant 1\right)$$

$$= \sum_{k=1}^{\infty} \left|b_{nk}^j\right|^r \left\{\left|b_{nk}^j\right|^{-r} P\left(\left|b_{nk}^j X\right| \geqslant 1\right)\right\} \to 0$$

uniformly in j, as $n \to \infty$.

Thus Y_n^j and Z_n^j are equivalent sequences. Therefore it is enogh to show that

$$P\left(\sum_{k=1}^{\infty} b_{nk}^j X_k > \delta\right) \leqslant \sigma^2\left(Z_n^j\right) \Big/ \delta^2 \to 0$$

as $n \to \infty$ uniformly as j.

Now

$$\sigma^2\left(Z_n^j\right) = \sum_{k=1}^{\infty} \sigma^2\left(X_{nk}^j\right) \leqslant \sum_{k=1}^{\infty} E\left(X_{nk}^j\right)^2$$

$$= \sum_{k=1}^{\infty} \left|b_{nk}^j\right|^2 \int_{\left|b_{nk}^j x\right| < 1} x^2 \, dP(X_k < x)$$

which by (11)

$$\leqslant 2 \sum_{k=1}^{\infty} \left|b_{nk}^{j}\right|^{2} \int_{0}^{\left|b_{nk}^{j}\right|^{-1}} x\, P(|X| \geqslant x)\, dx$$

$$= 2 \sum_{k=1}^{\infty} \left|b_{nk}^{j}\right|^{r} \left\{ b_{nk}^{j} \right\}^{2-r} \int_{0}^{\left|b_{nk}^{j}\right|^{-1}} xP(|X| \geqslant x)\, dx. \qquad \ldots(15)$$

By (14), the expression in (15) converges to 0 as $n \to \infty$ uniformly in j by virtue of the fact that

$$\left|b_{nk}^{j}\right|^{2-r} \int_{0}^{\left|b_{nk}^{j}\right|^{-1}} xP(|X| \geqslant x)\, dx \to 0 \qquad \ldots(16)$$

as $n \to \infty$, uniformly in j. (16) follows from [see Rohatgi[7]]

$$T^{r-2} \int_{0}^{T} x\, P(|X| \geqslant x)\, dx$$

$$\leqslant T^{r-2} \{ \int_{0}^{A} x\, P(|X| \geqslant x)\, dx + \epsilon \int_{0}^{T} x^{1-r}\, dx \} \to 0 \text{ as } T \to \infty.$$

This completes the proof.

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