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ENTROPIC MEASURE OF ECONOMIC INEQUALITY

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More than a dozen new measures of economic inequality based on measures of entropy and directed divergence are obtained and compared with earlier measures.

1. A GENERAL ENTROPIC MEASURE OF INEQUALITY

Let $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ be any probability distributions for which $p_i = 0$ whenever $q_i = 0$. Let $D(P : Q)$ be a measure of directed divergence of P from Q satisfying the following properties :

(i) $D(P : Q) \geqslant 0$ (ii) $D(P : Q) = 0$ iff $P = Q$

(iii) $D(P : Q)$ is a convex function of both $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$

(iv) $D(P : Q)$ does not change when the n pairs (p_i, q_i), $i = 1, 2, \ldots n$, are permuted among themselves.

Let U be uniform distribution $(1/n, 1/n, \ldots, 1/n)$ and let V denote any distribution in which any one of the probabilities is unity and all other probabilities are zero.

Let it be possible to write

$$D(P : U) = H(U) - H(P) \qquad \ldots(1)$$

where $H(P) \geqslant 0$, then it follows that

(i) $H(P) \leqslant H(U)$ (ii) $H(P) = H(U)$ iff $P = U$

(iii) $H(P)$ is a concave or pseudo-concave function of P

(iv) $H(P)$ does not change when $p_1, p_2, \ldots, p_n$ are permuted among themselves.

Now define

$$I(P) = \frac{H(U) - H(P)}{H(U)} \qquad \ldots(2)$$

then

$$I(U) = 0, I(V) = 1t \qquad \ldots(3)$$

Thus $I(P)$ vanishes when all the probabilities are equal i. e. when there is perfect equality. Again $I(P) = 1$ when one of the probabilities is unity and the rest are all zero i. e. when there is perfect inequality. For other distributions $I(P)$ lies between zero and unity. Thus $I(P)$ can be used as a general measure of inequality and we get a measure of inequality corresponding to every measure of directed divergence which can be decomposed in the form (1).

2. Some Special Measures of Economic Inequality

Let y_i be the income of the ith individual and let

$$\lambda = \frac{1}{n} \sum_{i=1}^{n} y_i \qquad \ldots(4)$$

be the mean income, then a measure of income inequality is given by

$$I = \frac{H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) - H\left(\frac{y_1}{n\lambda}, \frac{y_2}{n\lambda}, \ldots, \frac{y_n}{n\lambda}\right)}{H\left(\frac{1}{n}, \frac{1}{n}, , \ldots, \frac{1}{n}\right)}. \qquad \ldots(5)$$

Case I—This corresponds to Kullback-Leibler's[10] measure of directed divergence viz.

$$D_1(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \qquad \ldots(6)$$

so that

$$D_1(P:U) = \sum_{i=1}^{n} p_i \ln n p_i = \ln n - \left(-\sum_{i=1}^{n} p_i \ln p_i\right) \qquad \ldots(7)$$

$$= H_1(U) - H_1(P) \qquad \ldots(8)$$

where

$$H_1(P) = -\sum_{i=1}^{n} p_i \ln p_i \qquad \ldots(9)$$

in Shannons[13] measure of entropy. From (4), (5) and (9)

$$I_1 = \frac{\ln n + \sum_{i=1}^{n} \frac{y_i}{n\lambda} \ln \frac{y_i}{n\lambda}}{\ln n} = \frac{1}{n \ln n} \sum_{i=1}^{n} \frac{y_i}{\lambda} \ln \frac{y_i}{\lambda} \quad ...(10)$$

$$= \frac{1}{n \ln n} \sum_{i=1}^{n} \frac{ny_i}{\sum_{i=1}^{n} y_i} \ln \frac{ny_i}{\sum_{i=1}^{n} y_i}. \quad ...(11)$$

This is one of the measures obtained by Chakarvorty[3]. It lies between 0 and 1 and is unchanged when all incomes are multiplied by the same factor.

Case II—This corresponds to Havrda and Charvat's[5] measure of directed divergence viz.

$$D_2(P:Q) = \frac{1}{\alpha - 1} \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1 \right], \alpha \neq 1 \quad ...(12)$$

so that

$$D_2(P:V) = \frac{n^{\alpha-1}}{1-\alpha} \left[n^{1-\alpha} - 1 \right] - \frac{n^{\alpha-1}}{1-\alpha} \left[\sum_{i=1}^{n} p_i^{\alpha} - 1 \right] \quad ...(13)$$

$$= H_2(U) - H_2(P) \quad ...(14)$$

where

$$H_2(P) = \frac{n^{\alpha-1}}{1-\alpha} \left[\sum_{i=1}^{n} p_i^{\alpha} - 1 \right] \quad ...(15)$$

corresponds to Havrda and Charvat's[5] measure of entropy, so that

$$I_2 = 1 - \frac{\sum_{i=1}^{n} p_i^{\alpha} - 1}{n^{1-\alpha} - 1} = 1 - \frac{\sum_{i=1}^{n} \left(\frac{y_i}{n\lambda} \right)^{\alpha} - 1}{n^{1-\alpha} - 1}$$

(*equation continued on p.* 276)

$$= \frac{1}{n^{1-\alpha} - 1} \left[n^{1-\alpha} - \sum_{i=1}^{n} \left(\frac{y_i}{n\lambda} \right)^{\alpha} \right]$$

$$= \frac{1}{1 - n^{\alpha-1}} \left[1 - \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i}{\lambda} \right)^{\alpha} \right] \qquad \ldots(16)$$

which is the second measure of inequality obtained by Chakarvorty[3]. This also lies between 0 and 1 and is scale-invariant.

Case III—This corresponds to Renyi's[11] measure of directed divergence viz.

$$D_3(P:Q) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i^{\alpha} \; q_i^{1-\alpha}, \; \alpha \neq 1 \qquad \ldots(17)$$

so that

$$D_3(P:U) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i^{\alpha} \, n^{1-\alpha}$$

$$= H_3(U) - H_3(P) \qquad \ldots(18)$$

where

$$H_3(P) = \frac{1}{1 - \alpha} \ln \sum_{i=1}^{n} p_i^{\alpha} \qquad \ldots(19)$$

so that

$$I_3 = 1 - \frac{1}{(1 - \alpha) \ln n} \ln \sum_{i=1}^{n} \left(\frac{y_i}{\frac{n}{\sum_{i=1} y_i}} \right)^{\alpha} \qquad \ldots(20)$$

This also lies between 0 and 1 and is scale-invariant. It is easily verified that both (21) and (16) approach (11) as $\alpha \to 1$.

This is a new measure, but it is not of the form

$$A\,[n\,U(\lambda) - \sum_{i=1}^{n} U(y_i)\,]. \qquad \ldots(21)$$

Case IV—This corresponds to Kapur's[9] measure of directed divergence viz.

$$D_4(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{1}{a}\sum_{i=1}^{n} (1 + ap_i) \ln \frac{1 + a\,p_i}{1 + a\,q_i} \qquad \ldots(22)$$

so that

$$D_4(P:U) = \sum_{i=1}^{n} p_i \ln np_i - \frac{1}{a}\sum_{i=1}^{n} (1 + ap_i) \ln (1 + ap_i) + \frac{n + a}{a} \ln \frac{n + a}{n}$$

$$= H_4(U) - H_4(P) \qquad \ldots(23)$$

where

$$H_4(P) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}\sum_{i=1}^{n} (1 + ap_i) \ln (1 + ap_i) - \frac{1}{a}(1 + a) \ln (1 + a) \qquad \ldots(24)$$

and

$$H_4(U) = \ln n + \frac{n + a}{a} \ln \frac{n + a}{n} - \frac{1}{a}(1 + a) \ln (1 + a) \qquad \ldots(25)$$

so that

$$I_4 = \frac{\displaystyle\sum_{i=1}^{n} \frac{y_i}{n} \ln \frac{ny_i}{n} - \frac{1}{a}\sum_{i=1}^{n} \left(1 + \frac{ay_i}{n}\right) \ln \left(1 + \frac{ay_i}{n}\right) + \frac{n + a}{a} \ln \frac{n+a}{a}}{\ln n + \frac{n + a}{a} \ln \frac{n + a}{n} - \frac{1}{a}(1 + a) \ln (1+a)} \qquad \ldots(26)$$

where $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i$.

This also lies between 0 and 1, is scale-invariant and is of the form (21).

Case V—This coresponds to Kapur's[6,9] second measure of directed divergence i. e.

$$D_5(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{1}{a^2} \sum_{i=1}^{n} (1 + ap_i) \ln \frac{1 + ap_i}{1 + aq_i}. \qquad \ldots(27)$$

This gives

$$I_5 = \frac{\sum_{i=1}^{n} \frac{y_i}{n} \ln \frac{ny_i}{\sum_{i=1}^{n} y_i} - \frac{1}{a^2} \sum_{i=1}^{n} \left(1 + \frac{ay_i}{\sum_{i=1}^{n} y_i}\right) \ln \left(1 + \frac{ay_i}{\sum_{i=1}^{n} y_i}\right) + \frac{n+a}{a^2} \ln \frac{n+a}{n}}{\ln n + \frac{n+a}{a^2} \ln \frac{n+a}{n} - \frac{1}{a^2}(1+a) \ln (1+a)}. \qquad \ldots(28)$$

3. Another form for the Measure of Inequality

The five measures given in section 2 were based on the possibility of writing $D(P:U)$ in the form $H(U) - H(P)$. Even if this is not possible, we define

$$I = \frac{D(P:U)}{\max_P [D(P:U)]}. \qquad \ldots(29)$$

This lies between 0 and 1 and is scale-invariant. It may not be possible to write it in the form (21).

Case VI—This corresponds to Csiszer's[4] general measure of directed divergence

$$D(P:Q) = \sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right) \qquad \ldots(30)$$

where $\phi(\cdot)$ is convex twice-differentiable function. This gives

$$I = \frac{\sum_{i=1}^{n} \frac{1}{n} \phi(np_i)}{\frac{1}{n} \phi(n)} = \frac{\sum_{i=1}^{n} \phi(np_i)}{\phi(n)}$$

(*equation continued on p.* 279)

$$= \frac{\sum_{i=1}^{n} \phi\left(\frac{y_i}{\lambda}\right)}{\phi(n)} \qquad \ldots(31)$$

Case VI— When $\phi(x) = x \ln x$, I_6 reduces to I_1.

Case VII— When $\phi(x) = \frac{x^\alpha - 1}{\alpha - 1}$, we get

$$I_7 = \frac{\sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^\alpha - n}{n^\alpha - n} = \frac{1 - \frac{1}{n}\sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^\alpha}{1 - n^{\alpha-1}} = I_2. \qquad \ldots(32)$$

Case VIII—Another general class of measures of directed divergence is given by

$$D_8(P:Q) = \phi\left(\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}\right) \qquad \ldots(33)$$

where $\phi(.)$ is any monotonic increasing convex function. Corresponding to it, we have the measure

$$I_1 = \frac{\phi\left(\sum_{i=1}^{n} p_i \ln n\, p_i\right)}{\phi(\ln n)} = \frac{\phi\left(\sum_{i=1}^{n} \frac{y_i}{n} \ln \frac{ny_i}{\sum_{i=1}^{n} y_i}\right)}{\phi(\ln n)} - \qquad \ldots(34)$$

This always lies hetween 0 and 1, is scale-invariant but need not be of the form (21). If $\phi(x) = x$, we get case I. It is also a convex function of ϕ.

Case IX—Another general class of measures of directed divergence is given by

$$I_8(P:Q) = \frac{1}{\alpha - 1}\left[\phi\left(\sum_{i=1}^{n} p_i^\alpha \; q_i^{1-\alpha}\right) - \phi(1)\right], \alpha \neq 1 \qquad \ldots(35)$$

where $\phi(x)$ is again any convex monotonic increasing function of x. Corresponding to it, we have the measure

$$I_9 = \frac{\phi\left(\sum_{i=1}^{n} p_i^{\alpha}\ n^{\alpha-\alpha}\right) - \phi(1)}{\phi(n^{\alpha-1}) - \phi(1)}. \qquad \ldots(36)$$

This also satisfies all condition and we can get any number of special cases out of it.

4. Theil's Measures of Economic Inequality and their Generalisations

Let y_i be the income of the ith group and let $p_i = y_i / \sum_{i=1}^{n} y_i$. Again let z_i be the population of the ith group and let $q_i = z_i / \sum_{i=1}^{n} z_i$, then Theil[14,16] gave two measures of economic inequality viz.

$$D_{10}(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}, \quad D_{11}(P:Q) = \sum_{i=1}^{n} q_i \ln \frac{q_i}{p_i}. \qquad \ldots(37)$$

We can also construct a symmetric measure of directed divergence

$$D_{12}(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} + \sum_{i=1}^{n} q_i \ln \frac{q_i}{p_i}. \qquad \ldots(38)$$

Keeping q_i's fixed, we can construct the following normalised measure of inequality

$$I_{10} = \frac{\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}}{\ln \frac{1}{q_{\min}}} = \frac{\sum_{i=1}^{n} \frac{y_i}{\sum_{i=1}^{n} y_i} \ln \frac{y_i}{z_i} \frac{\sum_{i=1}^{n} z_i}{\sum_{i=1}^{n} y_i}}{\ln \sum_{i=1}^{n} z_i - \ln - \ln z_{\min}} \qquad \ldots(39)$$

where

$$q_{\min} = \min(q_1, q_2, \ldots, q_n), \quad z_{\min} = \min(z_1, z_2, \ldots z_n). \qquad \ldots(40)$$

From $D_{11}(P:Q)$ and $D_{12}(P:Q)$, we cannot construct normalised measure of inequality lying between 0 and 1 since for given q_i's and variation in p_i's, both these measures are unbounded. Similarly from any measure $D(P:Q)$ of directed divergence, we can construct the normalised measure of inequality

$$I_{13} = \frac{D(P:Q)}{\max_P D(P:Q)} = \frac{D\left(\frac{y_i}{n}, \frac{z_i}{n}\right)}{\max_{y_1, y_2, \cdots, y_n} D\left(\frac{y_i}{\sum_{i=1} y_i}, \frac{z_i}{\sum_{i=1} z_i}\right)} \quad \ldots(41)$$

provided $D(P:Q)$ is bounded for variations in P. We can also use

$$I_{14} = \frac{D(Q:P)}{\max_P D(Q:P)}, \quad I_{15} = \frac{D(P:Q) + D(Q,P)}{\max_P [D(P:Q) + D(Q,P)]} \quad \ldots(42)$$

provided the denominators are bounded. In particular, we can use

$$I_{15} = \frac{\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1}{q_{\min}^{1-\alpha} - 1} \qquad I_{16} = \frac{\phi\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}\right) - \varphi(1)}{\varphi\left(q_{\min}^{1-\alpha}\right) - \phi(1)} \quad \ldots(43)$$

$$I_{17} = \frac{\varphi\left(\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}\right)}{\varphi(\ln 1/q_{\min})} \quad \ldots(44)$$

where $\varphi(x)$ is any monotonic increasing convex function of x. Another measure we can use as

$$I_{18} = \frac{\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{1}{a}\sum_{i=1}^{n} (1 + a p_i) \ln \left(\frac{1 + a p_i}{1 + a q_i}\right)}{\ln \frac{1}{q_{\min}} - \frac{1}{a}(1 + a) \ln \frac{1 + a}{1 + a q_{\min}}}. \quad \ldots(45)$$

To see that the denominator of (45) gives the maximum value of its numerator for variation in p_i's, we note that the numerator in a convex function of $p_1, p_2 \ldots, p_n$ over the range

$$p_1 \geqslant 0,\ p_2 \geqslant 0,\ \ldots,\ p_n \geqslant 0\ p_1 + p_2 + \ldots + p_n = 1. \qquad \ldots(46)$$

As such its maximum value occurs at one of the corner points

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots) \ldots (0, 0, \ldots, 0, 1). \qquad \ldots(47)$$

The value at the ith corner point is

$$1/q_i - \frac{1}{a}(1 + a)\log(1 + a) + \frac{1}{a}(1 + a)\log(1 + a\,q_i). \qquad \ldots(48)$$

Now let

$$f(q) = -\ln q + \frac{1}{a}(1 + a)\log(1 + aq) \qquad \ldots(49)$$

so that

$$f'(q) = -\frac{1}{q} + \frac{1 + a}{1 + aq} = \frac{q - 1}{q\,(1 + aq)} < 0 \qquad \ldots(50)$$

and $f(q)$ is a decreasing function of q and attains its maximum value when q is minimum.

We can also use Kapur's[6,9] second measure of directed divergence with $1/a^2$ instead of $1/a$ in the numerator of (45). In this case however, instead of $f(q), f'(q)$ in (49), (50), we get

$$\varphi(q) = -\ln q + \frac{1}{a^2}(1 + a)\ln(1 + q)\ \varphi'(q) = -\frac{1}{q} + \frac{1}{a}\frac{1+a}{1+aq}$$

$$= \frac{q(1 - a^2) - a(1 - q)}{aq(1 + aq)}. \qquad \ldots(51)$$

If $a > 1$, $\varphi'(q) < 0$ and the maximum still occurs when $q = q_{\min}$, so that

$$I_{20} = \frac{\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{1}{a^2}\sum_{i=1}^{n}(1 + ap_i)\ln\frac{1 + ap_i}{1 + aq_i}}{\ln\frac{1}{q_{\min}} - \frac{1}{a^2}(1+a)\ln(1+a) + \frac{1}{a^2}(1+a)\ln(1+aq_{\min})};$$

$$a > 1 \qquad \ldots(52)$$

However if $a < 1$, $\varphi'(q) < 0$ when $q < a/(1 + a - a^2)$ and $\varphi'(q) > 0$ when $q > a/(1 + a - a^2)$, so that $\varphi(q)$ can be maximum when $q = q_{\min}$ or $q = q_{\max}$. Thus when $a < 1$, we shall replace the denominator in (50) by

$$\ln \frac{1}{q_{\max}} - \frac{1}{a^2}(1 + a)\ln(1 + a) + \frac{1}{a^2}(1 + a)\ln(1 + aq_{\max}) \qquad \ldots(53)$$

provided

$$\frac{1}{a^2}(1 + a)\ln\frac{1 + aq_{\max}}{1 + aq_{\min}} > \ln\frac{q_{\max}}{q_{\min}}. \qquad \ldots(54)$$

We can get other measures of inequality by using measures of directed divergence given by Adhikari and Joshi[1], Burbea and Rao[2], Kapur[7] and Roy and Turner[12].

5. Decomposition of Measure of Inequality

We have given a large number of measures of inequality each of which

(i) is zero in the case of perfect equality of income

(ii) is unity in the case of maximum inequality of income i. e. when the income is concentrated with one individual or one group.

(iii) lies between 0 and 1 in all other cases

(iv) is scale-invariant i. e. does not change when all incomes are multiplied by the same factor

(v) is unchanged when individuals (or groups) are interchanged among themselves

(vi) is a convex function of $p_1, p_2, \ldots, p_n$ where $p_i = y_i / \sum_{i=1}^{n} y_i$, but in general is not a convex function of $y_1, y_2, \ldots, y_n$.

Since we have a large number of measures, we can introduce additional conditions. Thus we may require that

(a) the measure should be capable of being extended smoothly to continuous income distribution case.

(b) the total inequality measure may be decomposable into an inequality measure between groups and an inequality measure within groups.

Theil[14,15] and Kapur[8] have shown that Kullback and Leibler[10] and Havrda and Charvat's[5] measure of directed divergence can be decomposed into two components

each, one corresponding to within group and the other corresponding to between groups. Thus let x_{ij} and y_{ij} be the incomes and populations of the jth subgroup of the ith group and let T_{il} and U_{il} be the total income and total population of the ith group and let T and U be total income and total population of all groups taken together, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{y_{ij}}{T} \ln \frac{y_{ij}/T}{z_{ij}/U} = \sum_{i=1}^{m} \frac{T_{i.}}{T} \sum_{j=1}^{n_i} \frac{y_{ij}}{T_{i.}} \ln \frac{y_{ij}/T_{i.}}{z_{ij}/U_i}$$

$$+ \sum_{i=1}^{m} \frac{T_{i.}}{T} \ln \frac{T_{i.}/T}{U_i/U}. \qquad \ldots(55)$$

In terms of normalised measure of inequality, this gives

$$I \ln \frac{U}{\min_{ij} z_{ij}} = \sum_{i=1}^{n} \frac{T_{i.}}{T} I_i \ln \frac{U_i}{\min_{j} z_{ij}} + I_B \ln \frac{U}{\min_{i} U_{i.}} \qquad \ldots(56)$$

where I is the overall measure of inequality, I_i is the measure of inequality within the ith group and I_B is the measure of inequality between groups.

Similarly for Havrda and Charvat's[5] measure, we get

$$I\left[\min_{ij}\left(\frac{z_{ij}}{U}\right)^{1-\alpha} - 1\right] = \sum_{i=1}^{m}\left(\frac{T_{i.}}{T}\right)^{\alpha}\left(\frac{U_{i.}}{U}\right)^{1-\alpha}$$

$$\times I_i\left[\min_{j}\left(\frac{z_{ij}}{U_{.}}\right)^{1-\alpha} - 1\right]$$

$$+ I_B\left[\min_{i}\left(\frac{U_{i.}}{U}\right)^{1-\alpha} - 1\right] \qquad \ldots(57)$$

If we divide(57) by $\alpha - 1$ and take the limit as $\alpha \to 1$, we shall get (56) from(57).

For most measures of inequality, such a decomposition into measures to inequality between and within groups is not possible and as such the above two measures get a certain preference.

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