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FIXED POINT THEOREMS FOR SET-VALUED MAPS ON A METRIC SPACE

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In this paper fixed point theorems for four set-valued mappings on a metric space are obtained : (i) when the mappings satisfy certain generalized contraction conditions and (ii) when the space is compact and the mappings satisfy certain contractive type conditions. Some of the results obtained generalize certain results of Brian Fisher[1-4] on set-valued as well as single-valued maps on metric spaces.

In this paper we obtain a number of fixed point theorems for four set-valued maps on a metric space, generalizing some of the results of Fisher[1-4] on set-valued and single-valued maps.

In Section 0 we give the basic terminology and the necessary notation and also provide the background needed for the results discussed in Sections 1 and 2.

In Section 1 we study the existence and uniqueness of fixed points for four set-valued maps on a metric space when the maps satisfy certain generalized contraction conditions.

In Section 2 we carry on the investigations of Section 1 to compact metric spaces by relaxing the generalized contraction conditions to the corresponding contractive-type conditions.

SECTION 0

Throughout this paper :

(X, d) is a metric space ;

$B(X)$ is the set of all non-empty bounded subsets of X;

A, B, C are non-empty subsets of X (unless otherwise stated);

F, G, S, T are mappings from X into $B(X)$;

$R(F) = \{Fx/x \in X\}$;

$F(A) = \bigcup_{x \in A} Fx$;

$D(A, B) = \sup \{d(x, y)/x \in A, y \in B\}$;

$D(a, B) = D(B, a) = \sup \{d(a, y)/y \in B\}$ $(a \in X)$;

$D(a, b) = d(a, b)$ $\quad (a, b \in X)$;

$d(a, B) = \inf \{d(a, y)/y \in B\}$;

$\rho(A, B) = \sup \{d(x, B)/x \in A\}$;

$H(A, B) = \max \{\rho(A, B), \rho(B, A)\}$;

$\bar{A}$ is the closure of A;

$A_\epsilon = \{y \in X/d(x, y) < \epsilon \text{ for some } x \text{ in } A\}$ $(\epsilon > 0)$;

R^+ is the set of all non-negative real numbers; and

$\phi : R^+ \to R^+$ is an increasing map with $\phi(t_+) < t \ \forall \ t \in (0, \infty)$.

It is known that

$D(A, B) = D(B, A)$;

$H(A, B) = H(B, A)$;

$D(A, C) \leqslant D(A, B) + D(B, C)$;

$\rho(A, C) \leqslant \rho(A, B) + \rho(B, C)$; and

$H(A, C) \leqslant H(A, B) + H(B, C)$.

Definition 0.1 (Fisher[1])—A sequence $\{A_n\}$ of subsets of X is said to converge to A if

(i) given $a \in A$ there is a sequence $\{a_n\}$ in X such that $a_n \in A_n \ \forall \ n = 1, 2, \ldots$ and $\{a_n\}$ converges to a; and

(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\epsilon \ \forall \ n \geqslant N$.

Remark (0.1): Condition (ii) of Definition 0.1 can be restated as: $\{\rho(A_n, A)\}$ converges to zero.

Lemma 0.1 (Fisher[1])—If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging respectively to A and B in $B(X)$ then the sequence $\{D(A_n, B_n)$ converges to $D(A, B)$.

Lemma 0.2—Let $\{A_n\}$ and $\{B_n\}$ be sequences in $B(X)$ such that $\{A_n\}$ converges to some A in $B(X)$ and $\{H(A_n, B_n)\}$ converges to zero. Then $\{B_n\}$ also converges to A.

PROOF : Let $a \in A$. Since $\{A_n\}$ converges to A, there exists a sequence $\{a_n\}$ in X such that $a_n \in A_n \ \forall \ n = 1, 2, \ldots$ and $\{a_n\}$ converges to a. Choose $b_n \in B_n$ such that $d(a_n, b_n) < \rho(A_n, B_n) + \frac{1}{n}$ $(n = 1, 2, \ldots)$. Since $\{H(A_n, B_n)\}$ converges to zero, $\{d(a_n, b_n)\}$ must converge to zero.

Hence $\{b_n\}$ converges to a.

We have

$$\rho(B_n, A_n) \leqslant \rho(B_n, A_n) + \rho(A_n, A) \leqslant H(A_n, B_n) + \rho(A_n, A) \ (n = 1, 2, \ldots).$$

Since $\{H(A_n, B_n)\}$ and $\{\rho(A_n, A)\}$ converge to zero, the sequence $\{\rho(B_n, A)\}$ converges to zero.

Hence $\{B_n\}$ coverges to A.

Proposition 0.1—Let $\{B_n\}$ be a sequence in $B(X)$ converging to some $B \in B(X)$. Then, for any a in X, the sequence $\{d(a, B_n)\}$ converges to $d(a, B)$.

PROOF : We have

$$d(a, B) \leqslant d(a, B_n) + \rho(B_n, B) \quad (n = 1, 2, \ldots).$$

Since $\{\rho(B_n, B)\}$ converges to zero, from the above inequality it follows that

$$d(a, B) \leqslant \lim_{n \to \infty} \inf d(a, B_n). \qquad \ldots(0.1)$$

Let $x \in B$. Then since $\{B_n\}$ converges to B, there exists a sequence $\{x_n\}$ in X such that $x_n \in B_n \ \forall \ n$ and $\{x_n\}$ converges to x.

We have

$$d(a, x_n) \geqslant d(a, B_n) \qquad (n = 1, 2, \ldots).$$

$$\therefore \lim_{n \to \infty} d(a, x_n) \geqslant \limsup_{n \to \infty} d(a, B_n).$$

$$\therefore d(a, x) \geqslant \limsup_{n \to \infty} d(a, B_n).$$

Since the above inequality holds for all x in B, we must have

$$d(a, B) \geq \limsup_{n\to\infty} d(a, B_n). \qquad \ldots(0.2)$$

From inequalities (0.1) and (0.2) it follows that $\{d(a, B_n)\}$ converges to $d(a, B)$.

Proposition 0.2—Let $\{B_n\}$ be a sequence in $B(X)$ converging to some $B \in B(X)$. Suppose that $\bar{A}$ is compact. Then the sequence $\{\rho(A, B_n)\}$ converges to $\rho(A, B)$.

PROOF : We have

$$\rho(A, B) \leq \rho(A, B_n) + \rho(B_n, B) \quad (n = 1, 2, \ldots).$$

Since $\{\rho(B_n, B)\}$ converges to zero, from the above inequality we have

$$\rho(A, B) \leq \liminf_{n\to\infty} \rho(A, B_n). \qquad \ldots(0.3)$$

We can choose a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ such that

$$\lim_{k\to\infty} \rho(A, B_{n_k}) = \limsup_{n\to\infty} \rho(A, B_n).$$

There exists a sequence $\{a_k\}$ in A such that

$$\rho(A, B_{n_k}) - \frac{1}{k} < d(a_k, B_{n_k}) \quad (k = 1, 2, \ldots).$$

Since $\bar{A}$ is compact, the sequence $\{a_k\}$ admits a convergent subsequence. Without loss of generality, we may assume that $\{a_k\}$ converges to some a in $\bar{A}$.

We have

$$d(a_k, B_{n_k}) \leq d(a_k, a) + d(a, B_{n_k}) \quad (k = 1, 2, \ldots).$$

Hence

$$\rho(A, B_{n_k}) - \frac{1}{k} < d(a_k, a) + d(a, B_{n_k}) \quad (k = 1, 2, \ldots).$$

In view of Proposition 0.1, the sequence $\{d(a, B_{n_k})\}$ converges to $d(a, B)$. Hence by taking limits on both sides of the above inequality as $k \to \infty$, we get

$$\lim_{k\to\infty} \rho(A, B_{n_k}) \leq d(a, B).$$

Hence

$$\lim_{n\to\infty} \sup\ \rho\ (A, B_n) \leqslant \rho\ (A, B). \qquad \ldots(0.4)$$

From inequalities (0.3) and (0.4) it follows that $\{\rho\ (A, B_n)\}$ converges to $\rho\ (A, B)$,

Remark 0.2 : Example 0.1 shows that the condition '$\bar{A}$ is compact' cannot be dropped from Proposition 0.2.

Example 0.1—Let $X = l^2$ with the usual norm on it.

Let u_n denote the sequence of real numbers whose nth term is one and the remaining terms are all zeros.

Let $A = \{u_k | k = 1, 2, .. \}$ and $A_n = \{u_1, u_2, \ldots, u_n\}$ $(n = 1, 2, \ldots)$.

Then $\{A_n\}$ converges to A and $\rho\ (A, A_n) = \sqrt{2}\ \forall\ n = 1, 2, \ldots$.

But $\rho\ (A, A) = 0$.

Hence $\{\rho\ (A, A_n)\}$ does not converge to $\rho\ (A, A)$.

Lemma 0.3—Let $\{A_n\}$ and $\{B_n\}$ be sequences in $B\ (X)$ converging respectively to A and B in $B\ (X)$. Suppose that $\bar{A}$ is compact. Then $\{\rho\ (A_n, B_n)\}$ converges to $\rho\ (A, B)$.

PROOF : We have

$$\rho\ (A_n, B_n) \leqslant \rho\ (A_n, A) + \rho\ (A, B_n) \quad (n = 1, 2, \ldots).$$

Since $\{\rho\ (A_n, A)\}$ converges to zero and by Proposition (0.2) $\{\rho\ (A, B_n)\}$ converges to $\rho\ (A, B)$, from the above inequality we have

$$\lim_{n\to\infty} \sup\ \rho\ (A_n, B_n) \leqslant \rho\ (A, B). \qquad \ldots(0.5)$$

We have

$$\rho\ (A, B) \leqslant \rho\ (A, A_n) + \rho\ (A_n, B_n) + \rho\ (B_n, B) \quad (n = 1, 2, \ldots).$$

Since $\{\rho\ (B_n, B)\}$ converges to zero and (by Proposition 0.2) $\{\rho\ (A, A_n)\}$ converges to $\rho\ (A, A)$ i. e. zero, from the above inequality it is clear that

$$\rho\ (A, B) \leqslant \lim_{n\to\infty} \inf\ \rho\ (A_n, B_n). \qquad \ldots(0.6)$$

From inequalities (0.5) and (0.6) it is evident that $\{\rho\ (A_n, B_n)\}$ converges to $\rho\ (A, B)$.

Definition 0.2 (Fisher[1])—F is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point of X.

Remark 0.2 : If F is continuous at $z \in X$ and $\epsilon > 0$, then $\exists\ \delta > 0$ such that $Fx \subseteq (Fz)_\epsilon$ whenever $x \in X$ and $d(x, z) < \delta$.

Lemma 0.4— Let F be continuous at $z \in X$ and $\{A_n\}$ be a sequence in $B(X)$ converging to $\{z\}$. Then $\{FA_n\}$ converges to Fz.

PROOF : Let $\{z_n\}$ be a sequence in X such that $z_n \in A_n\ \forall\ n = 1, 2, \ldots$. Then $\{z_n\}$ converges to z.

By the continuity of F at z, $\{Fz_n\}$ converges to Fz.

Hence for each $w \in Fz$ there corresponds a sequence $\{w_n\}$ in X such that $w_n \in Fz_n \subseteq FA_n\ \forall\ n = 1, 2, \ldots$ and $\{w_n\}$ converges to w.

Now let $\epsilon > 0$.

Since F is continuous at z, in view of Remark (0.2), $\exists\ \delta > 0$ such that

$$Fx \subseteq (Fz)_\epsilon \text{ whenever } x \in X \text{ and } d(x, z) < \delta.$$

Since $\{A_n\}$ converges to $\{z\}$, there exists a positive integer N such that

$$A_n \subseteq \{x \in X / d(x, z) < \delta\}\ \forall\ n \geqslant N.$$

Hence

$$FA \subseteq (Fz)_\epsilon\ \forall\ n \geqslant N.$$

Hence $\{FA_n\}$ converges to Fz.

Lemma 0.5—Suppose that $\lim_{t\to\infty} [t - \phi(t)] = \infty$, and $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative real numbers such that

$$a_n \leqslant \phi(a_n + b_n) \quad (n = 1, 2, \ldots) \qquad \ldots(0.7)$$

and $\{b_n\}$ converges to zero. Then $\{a_n\}$ converges to zero.

PROOF : From inequality (0.7) we have

$$(a_n + b_n) - \phi(a_n + b_n) \leqslant b_n \quad (n = 1, 2, \ldots).$$

Since $\lim_{t\to\infty} [t - \phi(t)] = \infty$ and $\{b_n\}$ is bounded, from the above inequality it follows that $\{a_n + b_n\}$ is bounded. In particular, $\{a_n\}$ is bounded.

If $\{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}$ with limit l, then by taking limits on both sides of inequality (0.7) as $n \to \infty$ along the sequence $\{n_k\}$ we get

$$l \leqslant \phi(l+).$$

Since $\phi(t+) < t \ \forall \ t \in (0, \infty)$, we must have $l = 0$.

Hence $\{a_n\}$ must converge to zero.

The concept of weak commutativity for self-maps on a metric space was introduced by Sessa[6]. We extend this notion to set-valued maps as follows :

Definition 0.3—We say that F and S commute weakly or F and S are weakly commutative or (F, S) is a weakly commuting pair (w. c. p.) if

$$H(FSx, SFx) \leqslant D(Sx, Fx) \ \forall \ x \in X.$$

Remark 0.3 : Commutativity implies weak commutativity. But the converse need not be true even in the case of single valued maps.

Definition 0.4 (Kaulgud and Pai[5])—A point x in X is said to be a fixed point of F if $x \in Fx$.

SECTION 1

Theorem 1.1—Suppose that

$$D(Fx, Gy) \leqslant \phi(\max\{D(Sx, Ty), D(Sx, Fx), D(Ty, Gy), D(Fx, Ty), D(Sx, Gy)\})...(1.1)$$

for all x, y in X and that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\{Fx_n\}$, $\{Sx_n\}$, $\{Ty_n\}$ and $\{Gy_n\}$ all converge to the same limit $\{z\}$ for some z in X. Then the following statements hold :

1. If $Sw = \{w\} = Fw$ then $w = z$.

2. If F is continuous at z, Fz is a singleton and (F, S) is a w. c. p. then $Fz = \{z\}$.

3. If S is continuous at z, (F, S) is a w. c. p. and Sz is a singleton then $Sz = \{z\} = Fz$. If further, $\lim_{t \to \infty} [t - \phi(t)] = \infty$, then F is continuous at z.

4. If $Sw = \{z\}$ for some w in X, (F, S) is a w. c. p. and either Fz or Sz is a singleton then $Fz = \{z\} = Sz$.

5. Statements 1 to 4 remain valid when F and S are replaced by G and T respectively.

6. If $F = G$, $S = T$ and $Sw = \{w\} \subseteq Fw$ then $w = z$.

7. If $F = G$, F is continuous at z, and (F, S) and (F, T) are w. c p.'s then $Fz = \{z\}$.

8. If $S = T$, S is continuous at z, and (F, S) and (G, S) are w. c. p.'s then $Fz = Sz = Gz = \{z\}$.

9. If $S = T$, $Sw = \{z\}$ for some w in X, and (F, S) and (G, S) are w. c. p.'s then $Fz = Sz = Gz = \{z\}$.

10 If $S = T$, S is continuous at z, Sz is a singleton and either (F, S) or (G, S) is a w. c. p. then $Fz = Gz = Sz = \{z\}$.

PROOF : We note that the sequences $\{D(Sx_n, Fx_n)\}$ and $\{D(Ty_n, Gy_n)\}$ converge to zero since the sequences $\{Fx_n\}$, $\{Sx_n\}$, $\{Gy_n\}$ and $\{Ty_n\}$ converge to $\{z\}$.

If FA, SA, GB and $TB \in B(X)$ then from inequality (1.1) it is clear that

$$D(FA, GB) \leqslant \phi(\max\{D(SA, TB), D(SA, FA), D(TB, GB), D(FA, TB), D(SA, GB)\}). \quad ...(1.2)$$

Taking $B = \{y_n\}$ in inequality (1.2) and then takingl imits on both sides of the inequality as $n \to \infty$ we get, on making use of Lemma 0.1, that

$$D(FA, z) \leqslant (\phi(\max\{D(SA, z), D(SA, FA), D(FA, z)\}) +) \quad ...(1.3)$$

when

$$FA \in B(X).$$

In particular, we have

$$D(Fx, z) \leqslant \phi((\max\{D(Sx, z), D(Sx, Fx), D(Fx, z)\}) +) \quad ...(1.4)$$

for all x in X.

1 Suppose $Sw = \{w\} = Fw$. Taking $x = w$ in inequality (1.4) we get

$$D(w, z) \leqslant \phi(D(w, z) +).$$

Since $\phi(t +) < t \ \forall \ t \in (0, \infty)$, we must have $D(w, z) = 0$.

Hence $w = z$.

2. Suppose F is continuous at z. Then, by Lemma 0.4, the sequences $\{FFx_n\}$ and $\{FSx_n\}$ converge to Fz since the sequences $\{Fx_n\}$ and $\{Sx_n\}$ converge to $\{z\}$. Suppose (F, S) is a w. c. p.. Then

$$H(FSx_n, SFx_n) \leqslant D(Sx_n, Fx_n) \quad (n = 1, 2, \ldots).$$

Since $\{D(Sx_n, Fx_n)\}$ converges to zero and $\{FSx_n\}$ converges to Fz, it is now evident from Lemma 0.2 that $\{SFx_n\}$ also converges to Fz.

Taking $A = Fx_n$ in inequality (1.3) and then taking limits on both sides as $n \to \infty$, we get, on using Lemma 0.1, that

$$D(Fz, z) \leqslant \phi((\max\{D(Fz, z), D(Fz, Fz)\}) +).$$

Suppose now that Fz is a singleton.

Then $D(Fz, Fz) = 0$ and therefore the above inequality reduces to

$$D(Fz, z) \leqslant \phi(D(Fz, z) +)$$

so that $D(Fz, z) = 0$.

Hence $Fz = \{z\}$.

3. Supppose S is continuous at z. Then, by Lemma 0.4, the sequences $\{S\,Sx_n\}$ and $\{S\,Fx_n\}$ converge to Sz since the sequences $\{Sx_n\}$ and $\{Fx_n\}$ converge to $\{z\}$.

Suppose (F, S) is a w. c. p.. Then $\{H(F\,Sx_n, SFx_n)\}$ converges to zero since $\{D(Sx_n, Fx_n)\}$ converges to zero.

Now, by Lemma 0.2, $\{FSx_n\}$ converges to Sz since $\{SFx_n\}$ converges to Sz.

Taking $A = Sx_n$ in inequality (1.3) and then taking limits on both sides as $n \to \infty$, on making use of Lemma 0.1 we get

$$D(Sz, z) \leqslant \phi((\max\{D(Sz, z), D(Sz, z)\}) +).$$

Suppose now that Sz is a singleton. Then the above inequality reduces to

$$D(Sz, z) \leqslant \phi(D(Sz, z) +)$$

so that $Sz = \{z\}$.

Now taking $x = z$ in inequality (1.4) we get

$$D(Fz, z) \leqslant \phi(D(Fz, z) +)$$

so that $Fz = \{z\}$.

Let $\{z_n\}$ be a sequence in X converging to z. Since S is continuous at z and $Sz = \{z\}$, the sequence $\{Sz_n\}$ converges to $\{z\}$.

Taking $x = z_n$ in inequality (1.4) we observe that

$$D(Fx_n, z) \leqslant \phi(D(Fx_n, z) + D(Sx_n, z)) +) \quad n = 1, 2, \ldots).$$

Suppose now that $\lim_{t\to\infty}[t - \phi(t)] = \infty$. Then from Lemma 0.5 it follows that $\{D(Fx_n, z)\}$ converges to zero. Hence $\{Fx_n\}$ converges to $\{z\} = Fz$.

Hence F is continuous at z.

4. Suppose $Sw = \{z\}$ for some w in X. Taking $x = w$ in inequality (1.4) we get

$$D(Fw, z) \leqslant \phi(D(Fw, z) +)$$

so that $Fw = \{z\}$.

Now suppose that (F, S) is a w. c. p.. Then

$$H(FSw, SFw) \leqslant D(Fw, Sw)$$

so that $H(Fz, Sz) = 0$.

Hence $\overline{Fz} = \overline{Sz}$.

Suppose now that either Fz or Sz is a singleton.

Then $Fz = Sz$ and each is a singleton. In particular, $D(Sz, Fz) = 0$. Now taking $x = z$ in inequality (1.4) we get

$$D(Fz, z) \leqslant \phi(D(Fz, z) +)$$

so that $Fz = \{z\}$. Hence $Sz = \{z\}$.

5. Statement 5 can be established along the lines of the proofs of statements 1 to 4.

6. Suppose $F = G$, $S = T$ and $Sw = \{w\} \subseteq Fw$. Taking $x = y = w$ in inequality (I) we get

$$D(Fw, Fw) \leqslant \phi(D(w, Fw))$$

so that

$$D(w, Fw) \leqslant \phi(D(w, Fw)).$$

Hence $Fw = \{w\}$.

Now from statement *l* we have $w = z$.

7. Suppose $F = G$. Then $\{Fy_n\}$ converges to $\{z\}$ and $\{D(Ty_n, Fy_n)\}$ converges zero. Suppose F is continuous at z.

Then, by Lemma 0.4 the sequences $\{FFx_n\}$, $\{FSx_n\}$, $\{FFy_n\}$ and $\{FTy_n\}$ converge to Fz.

Suppose that (F, S) and (F, T) are w. c. p.'s. Then $\{H(FSx_n, SFx_n)\}$ and $\{H(FTy_n, TFy_n)\}$ converge to zero.

Hence, by Lemma 0.2, the sequences $\{SFx_n\}$ and $\{TFy_n\}$ converge to Fz, since the sequences $\{FSx_n\}$ and $\{FTy_n\}$ converge to Fz. Taking $A = Fx_n$ and $B = Fy_n$ in inequality (1.2) and then taking limits on both sides of the inequality as $n \to \infty$, we get on using Lemma 0.1, that

$$D(Fz, Fz) \leqslant \phi(D(Fz, Fz) +)$$

so that $D(Fz, Fz) = 0$.

Hence Fz is a singleton.

Now from statement 2 it follows that $Fz = \{z\}$.

8. Suppose $S = T$. Then $\{Sy_n\}$ converges to $\{z\}$ and $\{D(Sy_n, Gy_n)\}$ converges to zero. Suppose S is continuous at z.

Then, by Lemma (0.4), the sequences $\{SSx_n\}$, $\{SFx_n\}$, $\{SSy_n\}$ and $\{SGy_n\}$ converge to Sz.

Suppose now that (F, S) and (G, S) are w. c. p.'s. Then $\{H(FSx_n, SFx_n)\}$ and $\{H(GSy_n, SGy_n)\}$ converge to zero.

Hence, by Lemma (0.2), the sequences $\{FSx_n\}$ and $\{GSy_n\}$ converge to Sz.

Taking $A = Sx_n$ and $B = Sy_n$ in inequality (1.2) and then taking limits on both sides of the inequality as $n \to \infty$, we get on using Lemma 0.1, that

$$D(Sz, Sz) \leqslant \phi(D(Sz, Sz) +)$$

so that Sz is a singleton.

Now from statements 3 and 5 it follows that $Fz = Sz = Gz = \{z\}$.

9. Suppose $Sw = \{z\}$ for some w in X and (F, S) is a w. c. p.. Then, from the proof of statement 4, we have $\overline{Fz} = \overline{Sz}$. Now suppose that $S = T$ and (G, S) is a w.c.p.. Then, proceeding as in the proof of statement 4, we obtain $\overline{Gz} = \overline{Sz}$. Hence $D(Fz, Gz) = D(Fz, Sz) = D(Sz, Sz) = D(Sz, Gz)$.

Now taking $x = y = z$ in inequality (1.1) we get

$$D(Sz, Sz) \leqslant \phi(D(Sz, Sz))$$

so that Sz is a singleton.

Now from statements 4 and 5 we have

$Fz = Sz = Gz = \{z\}$.

Statement 10 is evident from statements 3 and 5 and the observation that $Fz = \{z\}$ if $Sz = \{z\}$ (from inequality (1.4)) and $Gz = \{z\}$ if $Tz = \{z\}$.

Theorem 1.2—Suppose that (X, d) is complete, inequality (1.1) holds for all x, y in X and that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Sx_{n+1} \subseteq Fx_n = Gy_n \supseteq Ty_{n+1} \quad (n = 0.\ 1.\ 2,\ldots). \qquad \ldots(1.5)$$

Suppose also that $\lim_{t\to\infty}[t - \phi(t)] = \infty$. Then there exists a unique z in X such that $\{Fx_n\}$ converges to $\{z\}$. Further statements 1 to 10 of Theorem (1.1) hold here also.

PROOF : Let $A_n = Fx_n = Gy_n$ $(n = 0, 1, 2, \ldots)$.

For $i \geqslant 1$ and $j \geqslant 1$, taking $x = x_i$ and $y = y_j$ in inequality (1.1) and noting that $D(A, B) \leqslant D(A^1, B^1)$ when $A \subseteq A^1 \subseteq X$ and $B \subseteq B^1 \subseteq X$, it can be seen that

$$D(A_i, A_j) \leqslant \phi(\max\{D(A_{i-1}, A_{j-1}), D(A_{i-1}, A_i), D(A_{j-1}, A_j), D(A_i, A_{j-1}), D(A_{i-1}, A_j)\}). \qquad \ldots(1.6)$$

Let $\beta_n = \max\{D(A_i, A_j)/0 \leqslant i, j \leqslant n\}$ $(n = 0, 1, 2, \ldots)$

and $\alpha = \max\{D(A_0, A_0), D(A_0, A_1)\}$.

Since $D(A_0, A_j) \leqslant D(A_0, A_1) + D(A_1, A_j)$ from inequality (1.6) it is clear that

$$\beta_n \leqslant \phi(\beta_n) + \alpha \qquad (n = 0, 1, 2, \ldots).$$

Since $\lim_{t\to\infty}[t - \phi(t)] = \infty$, from the above inequality it follows that $\{\beta_n\}$ is bounded.

Hence $\{D(A_i, A_j)/i, j \geqslant 0\}$ is bounded.

Let $\gamma_n = \sup\{D(A_i, A_j)/\ i, j \geqslant n\}$ $(n = 0, 1, 2, \ldots)$.

The sequence $\{\gamma_n\}$ being a decreasing sequence of non-negative real numbers converges to some non-negative real number γ. We show that $\gamma = 0$.

From inequality (1.6) it is clear that

$$\gamma_n \leqslant \phi(\gamma_{n-1}) \ (n = 1, 2, \ldots).$$

Taking limits on both sides of the above inequality an $n \to \infty$, we get

$$\gamma \leqslant \phi(\gamma +)$$

so that $\gamma = 0$.

Let $\{z_n\}$ be a sequence in X such that $z_n \in A_n \ \forall \ n = 0, 1, 2, \ldots$.

Then $\{z_n\}$ is Cauchy, since $\{\gamma_n\}$ converges to zero and

$$d(z_i, z_j) \leqslant D(A_i, A_j) \ (i, j \geqslant 0).$$

ince (X, d) is complete, $\{z_n\}$ converges to some z in X.

Then $\{D(A_n, z)\}$ converges to zero, since

$$D(A_n, z) \leqslant D(A_n, z_n) + d(z_n, z) \leqslant D(A_n, A_n) + d(z_n, z)$$
$$(n = 0, 1, 2, \ldots).$$

If $w \in X$ is such that $\{D(A_n, w)\}$ converges to zero, then since

$$d(w, z) \leqslant D(w, A_n) + D(A_n, z) \to 0 \text{ as } n \to \infty,$$

we must have $w = z$. The rest of the theorem is evident from Theorem (1.1)

Corollary 1.1— Suppose that (X, d) is complete,

$$D(Fx, Fy) \leqslant \phi(\max\{D(Sx, Sy), D(Sx, Fx), D(Sy, Fy), D(Fx, Sy),$$
$$D(Sx, Fy)\})$$

for all x, y in X and that there is a sequence $\{x_n\}$ in X such that

$$Sx_{n+1} \subseteq Fx_n \ (n = 0, 1, 2, \ldots).$$

Suppose also that $\lim_{t \to \infty} [t - \phi(t)] = \infty$. Then there exists a unique z in X such that $\{Fx_n\}$ converges to $\{z\}$. Further, the following statements hold :

1. If $Sw = \{w\} \subseteq Fw$ then $w = z$.
2. If F is continuous at z and (F, S) is a w. c. p. then $Fz = \{z\}$.
3. If S is continuous at z and (F, S) is a w. c. p. then $Sz = \{z\} = Fz$ and F is continuous at z.

4. If $Sw = \{z\}$ for some w in X and (F, S) is a w. c. p. then $Sz = \{z\} = Fz$.

Remark 1.1 : Corollary (1.1) is an improvement over Theorem 1 of Fisher[1], Theorem 2 of Fisher[2] and the Theorem of Fisher[4].

By dropping the terms $D(Sx, Fx)$ and $D(Ty, Gy)$ from the right hand side of inequality (1.1), the restrictions imposed on the sequences $\{x_n\}$ and $\{y_n\}$ in Theorem 1.2 can be relaxed considerably. In fact, we have the following

Theorem 1.3 – Suppose that (X, d) is complete,

$$D(Fx, Gy) \leqslant \phi(\max\{D(Sx, Ty),\ D(Fx, Ty),\ D(Sx, Gy)\}) \qquad \ldots(1.7)$$

for all x, y in X and that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Sx_{n+1} \subseteq Fx_n \text{ and } Ty_{n+1} \subseteq Gy_n \quad (n = 0, 1, 2, \ldots). \qquad \ldots(1.8)$$

Suppose also tha t $\lim_{t \to \infty} [t - \phi(t)] = \infty$. Then there exists a unique z in X such that $\{Fx_n\}$ and $\{Gy_n\}$ converge to $\{z\}$.

Further, the following staements hold :

1. If $Sw = \{w\} \subseteq Fw$ then $w = z$ and $Fz = \{z\}$.
2. If F is continuous at z and (F, S) is a w. c. p. then $Fz = \{z\}$.
3. If S is continuous at z and (F, S) is a w. c. p. then $Sz = \{z\} = Fz$ and F is continuous at z.
4. If $Sw = \{z\}$ for some w in X and (F, S) is a w. c. p. then $Sz = \{z\} = Fz$.
5. Statements 1 to 4 remain valid if F and S are replaced by G and T respectively.

PROOF : Fix $y \in X$.

Let $\beta_n = \max\{D(Fx_i, Gy)/0 \leqslant i \leqslant n\}$ $(n = 0, 1, 2, \ldots)$,

and $\alpha = D(Ty, Gy)$.

For $i > 1$, from inequality (1.7) it can be seen that

$$D(Fx_i, Gy) \leqslant \phi(\max\{D(Fx_{i-1}, Ty)\ (D(Fx_i, Ty, D, Fx_{i-1}, Gy)\}).$$

Hence

$$\beta_n \leqslant \phi(\beta_n + \alpha) + \beta_0 \qquad (n = 0, 1, 2, \ldots).$$

i.e.

$$(\beta_n + \alpha) - \phi(\beta_n + \alpha) \leqslant \beta_0 \qquad (n = 0, 1, 2, \ldots).$$

Since $\lim_{t \to \infty} [t - \phi(t)] = \infty$, from the above inequality it follows that $\{\beta_n\}$ is bounded. Thus $\{D(Fx, Gy)\}$ is bounded for each y in X.

Similarly, it can be shown that the sequence $\{D(Fx, Gy_n)\}$ is bounded for each x in X. Hence

$$\sup \{D(Fx_i, Gy_j)/i, j \geqslant 0\} < +\infty.$$

Let $\gamma_n = \sup \{D(Fx_i, Gy_j)/i, j \geqslant n\}$ $(n = 0, 1, 2, \ldots)$.

The sequence $\{\gamma_n\}$ being a decreasing sequence of non-negative real numbers converges to some non-negative real number γ. We show that $\gamma = 0$.

For $i \geqslant 1$ and $j > 1$, from inequality (1.7) it can be seen that $D(Fx_i, Gy_j) \leqslant \phi(\max \{D(Fx_{i-1}, Gy_{j-1}), D(Fx_i, Gy_{j-1}), D(Fx_{i-1}, Gy_j)\})$ so that we have

$$\gamma_n \leqslant \phi(\gamma_{n-1}) \qquad (n = 1, 2, \ldots).$$

Taking limits on both sides of the above inequality as $n \to \infty$, we get

$$\gamma \leqslant \phi(\gamma +)$$

so that $\gamma = 0$.

Let $\{u_n\}$ and $\{v_n\}$ be sequences in X such that $u_n \in Fx_n$ and $v_n \in Gy_n \ \forall\ n = 1, 2, \ldots$.

Then we have

$$d(u_n, v_n) \leqslant D(Fx_n, Gy_n)$$

and

$$d(u_n, u_m) \leqslant D(Fx_n, Fx_m) \leqslant D(Fx_n, Gx_n) + D(FX_m, Gx_n).$$

In view of the above inequalities, the completeness of (X, d) and the convergence of $\{\gamma_n\}$ to zero we can conclude that $\{u_n\}$ and $\{v_n\}$ converge to the some limit z for same z in X. We have

$$D(Fx_n, z) \leqslant D(Fx_n, v_n) + d(v_n, z) \leqslant D(Fx_n, Gy_n) + d(v_n, z)$$

so that $\{D(Fx_n, z)\}$ converges to zero.

Similarly, it can be shown that the sequence $\{D(Gy_n, z)\}$ also converges to zero. The uniqueness of z is evident.

Statements 1 to 5 can be established by proceeding almost along the same lines as in the proofs of the corresponding statements of Theorem 1.1.

Corollary 1.2—Suppose that (X, d) is complete, and f, g, I and J are self-maps on X satisfying

$$d(fx, gy) \leqslant \phi(\max\{d(Ix, Jy), d(fx, Jy), d(Ix, gy)\})$$

for all x, y in X. Suppose also that $\lim_{t\to\infty}[t - \phi(t)] = \infty$ and that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ix_{n+1} = fx_n \text{ and } Jy_{n+1} = gy_n \ (n = 0, 1, 2, \ldots).$$

Then $\{fx_n\}$ and $\{gy_n\}$ are convergent with the same limit z for some z in X. Further, the following statements hold :

1. f and I have no common fixed point other than z.
2. If f is continuous at z and (f, I) is a w. c. p. then $fz = z$.
3. If I is continuous at z and (f, I) is a w. c. p. then $fz = z = Iz$.
4. If $Iw = z$ for some w in X and (f, I) is a w. c. p. then $fz = z = Iz$.
5. If $f(X) \subseteq I(X)$, f is continuous at z and (f, I) is a w. c. p. then $fz = z = Iz$.
6. Statements 1 to 5 remain valid if f and I are replaced by g and J respectively.

Remark 1.2 : Corollary 1.2 is an improvement over Theorems 1, 2, 3 and 4 of Fisher[3]. In fact, Corollary 1.2 shows that the boundedness condition on the metric in Theorem 3 of Fisher[3] and the finiteness condition on the fixed point sets of certain maps in Theorem 4 of Fisher[3] are redundant.

Remark 1.3 : Example 1.1 shows that the condition '$\lim_{t\to\infty}[t - \phi(t)] = \infty$' cannot be dropped from Corollary 1.2 even when $f = g$ and $I = J$ is the identity map. It also shows that the condition cannot be dropped from Corollary 1.1.

Example 1.1—Let $X = [1, \infty)$ with the usual metric.

Define $f : X \to X$ as $fx = 2x$ and $\phi : R^+ \to R^+$ as $\phi(t) = \dfrac{2t^2}{1 + 2t}$.

Then f is continuous on X: ϕ is an increasing continuous map on R^+ with $\phi(t) < t \ \forall\ t \in (0, \infty)$ and $\lim_{t\to\infty}[t - \phi(t)] = \frac{1}{2}$; and

$$|fx - fy| \leqslant \phi(\max\{|x-y|, |fx - y|, |x - fy|\}) \ \forall\ x, y \text{ in } X.$$

But f has no fixed point. In fact, for each x in X, the sequence $\{f^n x\}$ diverges to $+\infty$.

Theorem 1.4—Suppose that (X, d) is complete, inequality (1.1) holds for all x, y in X and that there are sequences $\{x_n\}$ and $\{A_n\}$ in X and $B(X)$ respectively such that

$$\left.\begin{aligned} Tx_{2n+1} &\subseteq Fx_{2n} = A_{2n}, \quad (n = 0, 1, 2, \ldots) \\ Sx_{2n+2} &\subseteq Gx_{2n+1} = A_{2n+1} \end{aligned}\right\} \qquad \ldots (1.9)$$

and $\{D(A_n, A_{n+1})\}$ converges to zero. Then there exists a unique z in X such that $\{A_n\}$ converges to $\{z\}$. Further, statements 1 to 10 of Theorem 1.1 hold here also.

PROOF : Claim—Given $\epsilon > 0$ there exists a positive integer N (depending on ϵ) such that $D(A_{2n}, A_{2m}) < \epsilon \ \forall\ n, m \geqslant N$.

Proof of the claim : Suppose the claim is false.

Then, since $\{D(A_n, A_{n+1})\}$ converges to zero, there exists a positive real number ϵ and strictly increasing sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$n_k < m_k, (D A_{2n_k}, A_{2m_k}) \geqslant \epsilon \text{ and } D(A_{2n_k}, A_{2m_k-2}) < \epsilon$$

for all $k = 1, 2, \ldots$.

Taking $x = x_{2n_k}$ and $Y = x_{2m_{k+1}}$ in inequality (1.1) it can be seen that

$$\begin{aligned} D(A_{2n_k}, A_{2m_k+1}) \leqslant \phi(\max \{&D(A_{2n_k-1}, A_{2m_k}), D(A_{2n_k-1}, A_{2n_k}), \\ &D(A_{2m_k}, A_{2m_k+1}), D(A_{2n_k}, A_{2m_k}), \\ &D(A_{2n_k-1}, A_{2m_k})\}). \end{aligned}$$

Taking limits on both sides of the above inequality as $k \to \infty$, we get

$$\epsilon \leqslant \phi(\epsilon +)$$

which is a contradiction since $\phi(t +) < t \ \forall\ t \in (0, \infty)$.

Hence the claim.

Since $\{D(A_n, A_{n+1})\}$ converges to zero, it is evident from the claim that $D(A_n, A_m) \to 0$ as both n and $m \to \infty$.

That there is a unique z in X such that $\{D(A_n, z)\}$ converges to zero can be shown exactly as in Theorem 1.2.

The rest of the theorem is evident from Theorem 1.1.

By slightly strengthening inequality (1.1) 'the condition $\{D(A_n, A_{n+1})\}$ converges to zero' can be dropped from Theorem 1.4. In fact, we have the following

Theorem 1.5—Suppose that (X, d) is complete,

$$D(Fx, Gy) \leqslant \phi(\max\{D(Sx, Ty),\ D(Sx, Fx),\ D(Ty, Fy),\ \alpha(x, y)\}) \qquad \ldots(1.10)$$

for all x, y in X, where

$$\alpha(x, y) = \begin{cases} \max\{D(Fx, Ty), D(Sx, Gy)\} & \text{if } Ty \not\subseteq Fx \text{ and } Sx \not\subseteq Gy \\ \frac{1}{2} D(Fx, Ty) & \text{if } Ty \not\subseteq Fx \text{ and } Sx \subseteq Gy \\ \frac{1}{2} D(Sx, Gy) & \text{if } Ty \subseteq Fx \text{ and } Sx \not\subseteq Gy \\ \frac{1}{2} \min\{D(Fx, Ty), D(Sx, Gy)\} & \text{if } Ty \subseteq Fx \text{ and } Sx \subseteq Gy. \end{cases}$$

Suppose also that there are sequences $\{x_n\}$ and $\{A_n\}$ in X and $B(X)$ respectively such that (1.9) holds. Then there exists a unique z in X such that $\{D(A_n, z)\}$ converges to zero.

Further, statements 1 to 10 of Theorem 1.1 hold.

PROOF : Taking $x = x_{2n}$ and $y = x_{2n+1}$ in inequality (1.10) we get

$$D(Fx_{2n}, Gx_{2n+1}) \leqslant \phi(\max\{D(Sx_{2n}, Tx_{2+1}), D(Sx_{2n}, Fx_{2n}),$$

$$D(Tx_{2n+1}, Gx_{2n+1}),\ \tfrac{1}{2} D(Sx_{2n}, Gx_{2n+1})\})$$

since $Tx_{2n+1} \subseteq Fx_{2n}$.

Hence

$$D(A_{2n}, A_{2n+1}) \leqslant \phi(\max D\{(A_{2n-1}, A_{2n}),\ D(A_{2n}, A_{2n+1})\}).$$

Taking $x = x_{2n}$ and $y = x_{2n-1}$ in inequality (1.10) it can be shown as above that

$$D(A_{2n}, A_{2n+1}) \leqslant \phi(\max\{D(A_{2n-1}, A_{2n-2}),\ D(A_{2n-1}, A_{2n})\}).$$

Thus we have

$$D(A_n, A_{n-1}) \leqslant \phi(\max\{D(A_{n-1}, A_n),\ D(A_n, A_{n+1})\})$$

for all $n = 1, 2, \ldots$.

Since $\phi(t+) < t\ \forall\ t \in (0, \infty)$, from the above inequality it follows that

$$D(A_n, A_{n+1}) \leqslant \phi(D(A_{n-1}, A_n)) \qquad (n = 1, 2, \ldots).$$

Hence $\{D(A_n, A_{n+1})\}$ is a decreasing sequence of non-negative real numbers and hence must converge to a non-negative real number γ. Taking limits on both sides of the above inequality as $n \to \infty$, we get

$$\gamma \leqslant \phi(\gamma +)$$

so that $\gamma = 0$.

Now the theorem is evident from Therem (1.4).

Corollary 1.3—Suppose that (X, d) is complete,

$$D(Fx, Gy) \leqslant \phi(\max, \{D(Sx, Ty), D(Sx, Fx), D(Ty\ Gy), \tfrac{1}{2}[\rho(Sx, Gy) + \rho(Ty, Fx)]\})$$

for all x, y in X and that there are sequences $\{x_n\}$ and $\{A_n\}$ in X and $B(X)$ respectively such that (1.9) holds. Then there exists a unique z in X such that $\{A_n\}$ converges to $\{z\}$. Further, statements 1 to 10 of Theorem 1.1 hold.

SECTION 2

Throughout this section we assume that (X, d) is compact; F, G, S and T are continuous on X; and (F, S) and (G, T) are weakly commuting pairs.

Theorem (2.1) — Suppose that

$$D(Fx, Gy) < \max\{D(Sx, Ty), D(Sx, Fx), D(Ty, Gy), D(Fx, Ty), D(Sx, Gy)\}\ldots(2.1)$$

for all those x,y in X for which the right hand side of the inequality is positive. Suppose also that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that (1.5) holds. If either (1) $S = T$ or (2) Sx and Tx are singletons for all x in X or (3) $F = G$ and $\mathscr{R}(F) \subseteq \mathscr{R}(S) \cup \mathscr{R}(T)$ or (4) Fx and Gx are singletons for all x in X and $\mathscr{R}(F) \cap \mathscr{R}(G) \subseteq \mathscr{R}(S) \cap \mathscr{R}(T)$, then there exists a unique z in X such that $Fz = Gz = Tz = Sz = \{z\}$. Further, if either $Sw = \{w\} = Fw$ or $Tw = \{w\} = Gw$ then $w = z$.

PROOF : *Case* (i)—There is a constant c in [0,1) such that

$$D(Fx, Gy) \leqslant c \max\{D(Sx, Ty), D(Sx, Fx), D(Ty, Gy), D(Fx, Ty). D(Sx, Gy)\}$$

for all x, y in X.

In this case, the theorem is evident from Theorem 1.2.

Case (ii)—Case (i) is false.

Then there exist an increasing sequences $\{c_n\}$ of real numbers in [0,1) increasing to 1 and sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$D(Fu_n, Gv_n) \geq c_n \max \{D(Su_n, Tv_n), D(Su_n, Fu_n), D(Tv_n, Gv_n), D(Fu_n, Tv_n), D(Su_n, Gv_n)\} \quad \ldots(2.2).$$

for all $n = 1, 2, \ldots$.

Since (X, d) is compact, the sequences $\{u_n\}$ and $\{v_n\}$ have convergent subsequences. Without loss of generality, we may assume that $\{u_n\}$ and $\{v_n\}$ are convergent with limits, say, u and v respectively.

Taking limits on both sides of inequality (2.2) as $n \to \infty$, we obtain, on making use of the continuity of F, G, S and T and Lemma (0.1), that

$$D(Fu, Gv) \geq \max \{D(Su, Tv), D(Su, Fu), D(Tv, Gv), D(Fu, Tv), D(Su, Gv)\}$$

so that the right hand side of the above inequality is zero.

Hence Fu, Su, Tv and Gv are all equal, each being equal to $\{z\}$ for some z in X.

Since (F, S) and (G, T) are w. c. p.'s, we must have

$$H(Fz, Sz) = 0 = H(Gz, Tz).$$

Hence $\overline{Fz} = \overline{Sz}$ and $\overline{Gz} = \overline{Tz}$.

Suppose that either $F = G$ or $S = T$ or Sx and Tx are singletons for all x in X or Fx and Gx are singletons for all x in X.

Then it is easy to see that

$$\max \{D(Sz, Tz), D(Sz, Fz), D(Tz, Gz), D(Fz, Tz), D(Sz, Gz)\} = D(Fz, Gz).$$

Hence the left hand side and therefore each side of the above equation must be zero. Hence Fz, Sz, Tz and Gz are all singletons and are equal.

It can now be seen that for $x = u$ and $y = z$ the left hand side and the right side of inequality (2.1) are equal so that each is zero. Hence $Gz = \{z\}$.

Hence $Fz = Gz = Tz = Sz = \{z\}$.

If $Fw = \{w\} = Sw$ then for $x = w$ and $z = y$ both sides of inequality (2.1) are equal so that each is zero. Hence $w = z$.

Similarly, if $Gw = \{w\} = Tw$ then also $w = z$.

Corollary 2.1—Suppose that

$$D(Fx, Fy) < \max \{D(Sx, Sy),\ D(Sx, Fx), D(Sy, Fy), D(Fx, Sy), D(Sx, Fx)\}$$

for all those x, y in X for which the right hand side of the inequality is positive. Suppose also that there is a sequence $\{x_n\}$ in X

such that

$$Sx_{n+1} \subseteq Fx_n \quad \forall\ n = 0, 1, 2, \ldots.$$

Then there exists a unique z in X such that $Fz = \{z\} = Sz$.

Remark 2.1 : Corollary (2.1) is an improvement over Theorem 2 of Fisher[1].

Theorem 2.2— Suppose that

$$D(Fx, Gy) < \max \{D(Sx, Ty), D(Fx, Ty), D(Sx, Gy)\}$$

for all those x, y in X for which the right hand side of the inequality is positive. Suppose also that there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that (1.8) holds. Then there exists a unique z in X such that $Fz = Sz = Tz = Gz = \{z\}$. Further, if either $Fw = Sw = \{w\}$ or $Gw = Tw = \{w\}$ then $w = z$.

PROOF : Similar to that of Theorem 2.1.

Remark 2.2 : Theorem (2.2) is an extension of Theorem 5 of Fisher[3] to set-valued maps, with some of the conditions in the latter being slightly relaxed.

Theorem 2.3—Suppose that

$$D(Fx, Gy) < \max \{D(Sx, Ty),\ D(Sx, Fx),\ D(Ty, Gy),\ \tfrac{1}{2}[\rho(Sx, Gy) + \rho(Ty, Fx)]\}$$

for all those x, y in X for which the right-hand side of the inequality in positive. Suppose also that there is a sequence $\{x_n\}$ in X such that

$$Tx_{2n+1} \subseteq Fx_{2n} \text{ and } Sx_{2n+2} \subseteq Gx_{2n+1} \text{ for all } n = 0, 1, 2, \ldots.$$

If either (1) $S = T$ or (2) Sx and Tx are singletons for all x in X or (3) $F = G$ and $\mathscr{R}(F) \subseteq \mathscr{R}(S) \cup \mathscr{R}(T)$ or (4) Fx and Gx are singletons for all x in X and $\mathscr{R}(F) \cap$

$\mathscr{R}(G) \subseteq \mathscr{R}(S) \cap \mathscr{R}(T)$, then there exists a unique z in X such that $Fz = Sz = Tz = Gz = \{z\}$.

Further, if either $Fw = Sw = \{w\}$ or $Gw = Tw = \{w\}$ then $w = z$.

PROOF : The Theorem can be established by proceeding almost along the same lines as in Theorem 2.1, of course, by invoking Lemma 0.3 at an appropriate stage.

Corollary 2.2—Let f, g, I and J be continuous self-maps on X. Suppose that (f, I) and (g, J) are w. c. p's and

$$d(fx, gy) < \max \{d\,Ix, Jy),\ d(Ix, fx),\ d(Jy, gy),\ \tfrac{1}{2}[d(fx, Jy) + d(Ix, gy)]\}$$

for all those x, y in X for which the right hand side of the inequality is positive.

Suppose also that there is a sequence $\{x_n\}$ in X such that

$$fx_{2n} = Jx_{2n+1} \text{ and } gx_{2n+1} = Ix_{2n+2} \qquad (n = 0, 1, 2, \ldots).$$

Then f, g, I and J have a unique common fixed point z. Further, z is the only common fixed point of f and I and also of g and J.

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REFERENCES

1. B. Fisher, *Rostock. Math Kolloq.* **18** (1981), 69–77.
2. B. Fisher, *Fundamenta Mathematica*, C XII (1981), 141–45.
3. B. Fisher, *J. Univ. Kuwait (Sci.)* **8** (1981), 131–38.
4. B. Fisher, *J. Univ. Kuwait (Sci)* **9** (1982), 175–79.
5. N. N. Kaulgud, and D. V. Pai, *Nieuw Arch. Wisk.* **23** (1975), 49–66.
6. S. Sessa, *Publ. Inst. Math.* **32** (46) (1982) 149–53.