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ON THE EQUALITY OF CURVATURE TENSORS IN SPECIAL FINSLER SPACES

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The purpose of the present paper is to study the equality of Berwald curvature tensor H_{ijkh} with Rund curvature tensor K_{ijkh} or Hashiguchi curvature tensor I_{ijkh} in various special Finsler Spaces.

1. PRELIMINARIES

Let F^n be an n-dimensional Finsler space with the fundamental function $L(x, y)$, the positive definite metric tensor $g_{ij} = \frac{1}{2}\,\dot\partial_i\,\dot\partial_j\,L^2$ and angular metric tensor $h_{ij} = g_{ij} - l_i\,l_j$ where $l_i = \dot\partial_i\,L$, $\dot\partial_i = \dfrac{\partial}{\partial y^i}$.

We use a notation $p.$ which is the projection on the indicatrix, e. g., for a tensor T^i_j of type (1, 1) $p.\ T^i_j = h^i_a\ h^b_j\ T^a_b$ where $h^i_a = \delta^i_a - l^i\,l_a$, $l^i = L^{-1}\,y^i$. Any tensor invariant by operator $p.$ is called indicatric tensor. Any tensor L_{ik} in F^n can be decomposed into the form

$$L_{ik} = m_{ik} + a_i\,l_k + b_k\,l_i + c\,l_i\,l_k \qquad \ldots(1.1)$$

where $m_{ik} = p.\ L_{ik}$, $a_i = L^{-1}\,p.\ L_{io}$, $b_k = L^{-1}\,p.\ L_{ok}$, $c = L^{-2}\,L_{oo}$ and index 0 denotes transvection by y.

For Cartan connection $C\Gamma$, h – and v-covariant derivatives of a Finsler tensor field X^i_j given by

$$X^i_{j|k} = \delta_k\,X^i_j + F^i_{mk}\,X^m_j - F^m_{jk}\,X^i_m$$

$$X^i_{j|k} = \dot\partial_k\,X^i_j + C^i_{mk}\,X^m_j - C^m_{jk}\,X^i_m$$

where $\delta_k = \partial_k - N^m_k\,\dot\partial_m$, $\partial_k = \partial/\partial x^k$ $N^i_k = F^i_{ok}$ F^i_{ik}. are connection coefficients with respect to $C\Gamma$, $C^i_{jk} = g^{im}\,C_{jmk}$ and $C_{jmk} = \frac{1}{2}\,\dot\partial_k\,g_{jm}$.

h — and v — curvature tensors of $C\Gamma$ are R^i_{hjk} and S^i_{hjk} and (*v*) h—, (*h*) hv—and (*v*) hv-torsion tensors of $C\Gamma$ are R^i_{jk}, C^i_{jk} and P^i_{jk} respectively. On the other hand if K^i_{hjk}, H^i_{hjk} and I^i_{hjk} denotes the h-curvature tensors of Rund, Berwald and Hashiguchi connections respectively. The relation between these curvature tensors and that of $C\Gamma$ are

$$K^i_{hjk} = R^i_{hjk} - C^i_{hr} \quad R^r_{jk} \qquad \ldots(1.2)$$

$$H^i_{hjk} = R^i_{hjk} - C^i_{hr} \quad R^r_{jk} + \left\{P^i_{hj|k} + P^i_{kr} \quad P^r_{jh} - j \mid k\right\} \qquad \ldots(1.3)$$

$$I^i_{hjk} = R^i_{hjk} + \left\{P^i_{jh|k} + P^i_{kr} \quad P^r_{jh} - j \mid k\right\} \text{ (Sinha[11])} \qquad \ldots(1.4)$$

where $j \mid i$ means interchange of indices i and j and subtraction.

A Finsler space F^n ($n > 2$) of scalar curvature K ($K \neq 0$) is characterized by[9]

$$H^i_k = L^2 \, K h^i_k \, . \qquad \ldots(1.5)$$

Specially, if K is constant, then F^n is said to be of constant cvrvature.

A Finsler space F^n ($n > 3$) satisfying[4]

$$R_{hijk} = g_{hj} \, M_{ik} + g_{ik} \, M_{hj} - j \mid k \qquad \ldots(1.6)$$

where

$$M_{ik} = (R_{ik} - \frac{r}{2} g_{ik})/(n-2), \; R_{hj} = R^r_{hjr}, \; r = R^i_{\cdot} \mid (n-1)$$

is called an $R3$-like Finsler space.

A Finsler space F^n ($n \geqslant 3$) with the non-zero length C of the torsion vector C^i is called semi-C-reducible[7], if the (*h*) hv-torsion tensor C_{ijk} is of the form

$$C_{ijk} = \frac{p}{(n+1)} \, (h_{ij} \, C_k + h_{jk} \, C_i + h_{ki} \, C_j) + \frac{q}{C^2} C_i \, C_j \, C_k \qquad \ldots(1.7)$$

where p and q ($= 1 - p$) do not vanish, p is called the characteristic scalar of F^n and $C^2 := C^i \, C_i$, $C_i := g^{jk} \, C_{ijk}$.

An h-isotropic Finsler space F^n $(n > 2)$ is characterized by[2]

$$R_{ijkm} = R\,(g_{ik}\,g_{jm} - g_{im}\,g_{jk}) \quad (R : \text{a constant}). \qquad ...(1.8)$$

A Finsler space F^n $(n > 2)$ satisfying the condition[13]

$$p.\ H_{hijk} = k\,(h_{hj}\ h_{ik} - h_{hk}\,h_{ij}) \qquad ...(1.9)$$

is called a Finsler space of Hp-scalar curvature and k is called Hp-scalar curvature.

Izumi[3] introduced an interesting tensor F^i_{hjk} and Sakaguchi[8] investigated a Finsler space F^n $(n > 2)$ with $F^i_{hjk} = 0$. This space is characterized by

$$H^i_{hjk} = L_{hj}\ \delta^i_k\ + g_{hj}\,L^i_k\ -\ h^i_h\ \ L_{jk} - j \mid k \qquad ...(1.10)$$

where

$$L_{hj} = [(n-1)\,H_{hj} - \tfrac{1}{2}\,g^{rs}\,H_{rs}\,g_{hj} + (H_{mh} - H_{hm})\,l^m\,l_j]/\,(n-1)\,(n-2)$$

$$L^i_k\ = g^{im}\,L_{mk},\ H_{hj}\ = H^m_{hjm}.$$

A non-Riemannian Finsler space F^n $(n > 2)$ is called C-reducible, if the (h) hv-torsion tensor C_{ijk} of F^n is written in the form[6]

$$C_{ijk} = \frac{1}{(n+1)}\,(h_{ij}\,C_k + h_{jk}\,C_i + h_{ki}\,C_j). \qquad ...(1.11)$$

A Finsler space of perpendicular scalar curvature is characterized by[5]

$$p.\ R_{hijk} = R\,h_{hj}\,h_{ik} + \tfrac{1}{2}\,\left(Z^m_{hj}\,C_{mik} + Z^m_{ik}\,C_{mhj} - j \mid k\right) \qquad ...(1.12)$$

where

$$Z^m_{hj} :=p.\ H^m_{hj}\ . \qquad ...(1.13)$$

2. On the Equality of Rund and Berwald h-Curvature Tensors

(Equivalently on the equality of Cartan and Hashiguchi h-curvature tensors.)

Definition 2.1—A Finsler space with equal Rund and Berwald h-curvature tensors will be called RBF^n-space.

Definition 2.2—A Finsler space with equal Cartan and Hashiguchi h-curvature tensors will be called CHF^n-space.

First we shall show :

Proposition 2.1—RBF^n-space is characterized by

$$\text{(a)} \quad P^l_{jk} \; P^l_{ik} - j \mid k = 0$$

and

$$\text{(b)} \quad T^l_{hjk} := P^l_{hj|k} - j \mid k = 0. \qquad \ldots(2.1)$$

PROOF : Using (1.2) and (1.3), the equality $K^l_{hjk} = H^l_{hjk}$ gives

$$P^l_{hj|k} + P^l_{hj} \; P^l_{ik} - j \mid k = 0$$

or

$$P_{hij|k} + P^l_{hj} \; P_{ilk} - j \mid k = 0. \qquad \ldots(2.2)$$

The skew-symmetric and symmetric parts with respect to indices h and i of (2.2) yields (2.1a) and (2.1b). The tensor T^l_{hjk} in (2.1b) is called a stretch-tensor. We are known the following

Lemma 2.1 (Shibata[10], Theorem 4)—If the stretch tensor of the space vanishes, the Finsler space of scalar curvature is a Riemannian space of constant curvature. Hence we have :

Theorem 2.1—RBF^n-space of scalar curvature is a Riemannian space of constant curvature provided $K \neq 0$.

In his paper, Akbar—Zadeh[1] proved the following :

Lemma 2.2—An h-isotropic Finsler space satisfies

$$\text{(a)} \quad R\left(P^l_{jkl} - P^l_{jik} \right) = 0$$

and

$$\text{(b)} \quad R\, S^l_{jki} = 0. \qquad \ldots(2.3)$$

Therefore we can understand that in an h-isotropic Finsler, it holds that $R^l_{jkl} = 0$ or S^l_{hjk}.

Theorem 2.2—If RBF^n-space is an h-isotropic Finsler space, the space satisfies $RC^l_{jk} = 0$.

PROOF : From (1.3) and (2.2), we find

$$H^l_{hjk} = R\left(g_{hj}\ \delta^l_k - g_{hk}\ \delta^l_j\right) - C^l_{hl}\ R^l_{jk}. \qquad ...(2.4)$$

Contracting (2.4) by y^h, we get

$$R^l_{jk} = R\left(y_j\ \delta^l_k - y_k\ \delta^l_j\right) \qquad ...(2.5)$$

and hence we have

$$H^l_{hjk} = \dot{\partial}_h\ R^l_{jk} = R\left(g_{hj}\,\delta^l_k - g_{hk}\ \delta^l_j\right). \qquad ...(2.6)$$

Thus, with (2.4), (2.5), (2.6) in mind, we obtain

$$C^l_{hl}\ R^l_{jk} = R\left(y_j\ C^l_{hk} - y_k\ C^l_{hj}\right) = 0$$

which implies the assertion.

Sakaguchi[8] showed the following :

Lemma 2.3 (Sakaguchi[8], Theorem 4.11)—If the stretch tensor of the space vanishes, the Finsler space with $F^l_{hjk} = 0$ is a Riemannian space of constant curvature or a Finsler space satisfying $S^l_{hjk} = 0$ for $C_l \neq 0$.

Hence, we can state

Theorem 2.3—In an RBF^n-space with $F^l_{hjk} = 0$, either hv-curvature tensor S^l_{hjk} of $C\Gamma$ vanishes for $C_l \neq 0$ or it is of Riemannian space of constant curvature $K \neq 0$.

Theorem 2.4—If an RBF^n-space is an $R3$—like Finsler space of Hp-scalar curvature, then it is a Riemannian space of constant curvature, provided $K \neq 0$.

PROOF : Sinha and Ram[12] has shown that an $R3$-like Finsler space of Hp-scalar curvature is of scalar curvature and perpendicular scalar curvature. Since the stretch curvature tensor of RBF^n-space vanishes, it becomes a Riemannian space of constant curvature on account of Lemma 2.1.

Theorem 2.5—IfRBF^n-space is semi-C-reducible with $p|_0 = 0$ then it reduces to C-reducible Finsler space or satisfies $\mu C^2 = \nu^2$ provided $\nu \neq 0$.

PROOF : From (1.7) the h-covariant derivative $C_{ijk|h}$ of C_{ijk} is written in the form

$$C_{ijk|h} = 0_{(ijk)} \left\{ h_{ij} (p_{|h} C_k + p C_{k|h})/ (n + 1) + \frac{q}{C^2} C_{i|h} C_j C_k \right\}$$

$$- (p_{|h}/C^2 + q C^2_{|h}/C^4) C_i C_j C_k \qquad \ldots(2.7)$$

where we have used the equation $h_{ijk} = 0$ and $0_{(ijk)}$ means the cyclic permutation of indices i, j, k and summation. Transvecting (2.7) by y^h and using $P_{ijk} = C_{ijk|0}$, $p_{|0} = 0$, we have

$$P_{ijk} = 0_{(ijk)} \left\{ \frac{p}{(n+1)} h_{ij} P_k + \frac{q}{C^2} P_i C_j C_k \right\} - \frac{2q\nu}{C^4} C_i C_j C_k \qquad \ldots(2.8)$$

where

$$P_i = C_{i|0} \text{ and } \nu = C^k P_k.$$

Taking h-covariant derivative $P_{ijk|h}$ of (2.8) and transvecting the result by y^h and using the fact $P_{hij|0} = 0$ in RBF^n-space, we have

$$q \left[0_{ijk} \left\{ \frac{1}{C^2} P_i (P_j C_k + P_k C_j) - \frac{4\nu}{C^4} P_i C_j C_k \right\} - \frac{2}{C^8} (\mu C^4 \right.$$

$$\left. - 4C^2\nu^2) \times C_i C_j C_k \right] = 0$$

where

$$\mu = P^k P_k.$$

From this equation it is clear that there are two possibilities : Either $q = 0$ or

$$0_{ijk} \left\{ \frac{1}{C^2} P_i (P_j C_k + P_k C_j) - \frac{4\nu}{C^4} P_i C_j C_k \right\}$$

$$- \frac{2}{C^8} (\mu C^4 - 4C^2\nu^2) C_i C_j C_k = 0. \qquad \ldots(2.9)$$

In case $q = 0$, we see from (1.7) and (1.11), the RBF^n-space is C-reducible.

On the other hand, if (2.9) holds, then transvection of (2.9) by C^i and P^j successively on simplification gives

$$(\mu C^2 - \nu^2)(C^2 P_k - 2\nu C_k) = 0. \qquad \ldots(2.10)$$

Transvecting (2.10) by C^k, we have

$$-\nu C^2 (\mu C^2 - \nu^2) = 0$$

which proves the second part of the Theorem.

Remark : All the results shown for RBF^n-space will also hold in CHF^n-space because $P_{hijk} - P_{hikj} = 0$ and $P^m_{hj} P_{mik} - j \mid k = 0$ in this case follows from (1.4).

3. On the Equality of Berwald and Hashiguchi h-Curvature Tensors

(Equivalently on the equality of Cartan and Rund h-curvature tensor.)

Definition 3.1—A Finsler space with equal Berwald and Hashiguchi h-curvature tensor will be called BHF^n-space.

Definition 3.2—A Finsler space with equal Cartan and Rund h-curvature tensors will be called CRF^n-space.

Lemma 3.1—The BHF^n-space and the CRF^n-spaces are characterized by

$$C^l_{hi} R^h_{jk} = 0. \qquad \ldots(3.1)$$

Proof : (1.2), (1.3) and (1.4) gives (3.1).

Remark : A Finsler space with (3.1) was first investigated by Soo's[14].

Theorem 3.1—If BHF^n-space is of scalar curvature, the space is a Riemannian space of constant curvature provided $K \neq 0$.

Proof : Finsler space of scalar curvature is charaterized by (1.5) or

$$R^i_{jk} = L(K l_j + \tfrac{1}{3} K_j) h^i_k - j \mid k, \; K_j := L \dot{\partial}_j K. \qquad \ldots(3.2)$$

Accordingly, using (3.1) and (3.2) we find

$$(K l_j + \tfrac{1}{3} K_j) C_{hik} - j \mid k = 0. \qquad \ldots(3.3)$$

Hence we have

$$K\, C_{hik} = 0$$

and

$$K_j\, C_{hik} - j \mid k = 0$$

which proves the assertion.

Theorem 3.2—In BHF^n-space, an $R3$-like C-reducible Finsler space is a Riemannian space of constant curvature.

PROOF : In an $R3$-like Finsler space (Izumi[3], (3.2a))

$$H^l_{jk} = R^l_{jk} = L\left[l_j \left(m^l_k + Ch^l_k \right) + b_j\, h^l_k \right] - j \mid k. \qquad ...(3.4)$$

From (3.1) and (3.4), we have

$$l_j \left[C^r_{hi}\, m_{rk} + c\, C_{hik} \right] + b_j\, C_{hik} - j \mid k = 0. \qquad ...(3.5)$$

Transvecting (3.5) by l^j, we have

$$C^r_{hi}\, m_{rk} + c\, C_{hik} = 0. \qquad ...(3.6)$$

Taking skew-symmetry in h and k, we have

$$C^r_{hi}\, m_{rk} - h \mid k = 0. \qquad ...(3.7)$$

Also, in an $R3$-like Finsler space (Izumi and Yoshida[4], 5.2).

$$L\, C^m_{hi}\, m_{mj} + (a_j - b_j)\, h_{hl} - i \mid j = 0. \qquad ...(3.8)$$

From (3.7) and (3.8), we have

$$(a_j - b_j)\, h_{hl} - i \mid j = 0. \qquad ...(3.9)$$

Transvecting (3.9) by h^{hi}, we see that $a_l = b_l$. Now from (1.1), we have

$$L_{lk} - L_{kl} = (a_l - b_l)\, l_k - i \mid k. \qquad ...(3.10)$$

From (3.9) and (3.10), we find $L_{lk} = L_{kl}$.

On the other hand, we know the following

Lemma 3.2 (Izumi and Yoshida[4], Theorem 5.3)—If $L_{ik} = L_{ki}$, a C-recucible $R3$-like Finsler space is a Finsler space of scalar curvature. Thus, in virtue of Theorem 3.1, the assertion is proved.

Theorem 3.3—In BHF^n-space

(i) A C-reducible Finsler space F^n $(n > 3)$ with $F^i_{hjk} = 0$ and $C_i \neq 0$ is of constant curvature and

(ii) A Finsler space with $F^i_{hjk} = 0$ is a Finsler space satisfying $S^i_{hjk} = 0$ for $C_i \neq 0$ or a Riemannian space of constant curvature provided $K \neq 0$.

PROOF : From (1.1) and (1.10), we obtain

$$R^i_{jk} = L\left\{ l_j\left(m^i_k + ch^i_k \right) + b_j\, h^i_k \right\} - j \mid k. \qquad ...(3.11)$$

Retreating the steps of the proof of Theorem 3.2, we have

$$L_{ik} = L_{ki}. \qquad ...(3.12)$$

Now first part of the Theorem follows from the following :

Lemma 3.3 (Sakaguchi[8], Theorem 4.7)—If $L_{ik} = L_{ki}$ and $C_i \neq 0$, a C-reducible Finsler space F^n $(n > 3)$ is a Finsler space of constant curvature.

Further, if F^n with $F^i_{hjk} = 0$ (Sakaguchi[8]), we have

$$(P_{hijk} + h_{hi}\, L_{jk} - j \mid k) - C_{hir}\, R^r_{jk} = 0. \qquad ...(3.13)$$

Using (3.1) and (3.12) in (3.13), we have

$$2T_{hijk} = P_{hijk} - P_{hikj} = 0. \qquad ...(3.14)$$

The second part of the Theorem follows from Lemma 2.3

Theorem 3.4—If BHF^n-space is an h-isotropic non-Riemannian then all the four h-curvature tensor vanishes.

PROOF : In BHF^n-space

$$C_{hir}\, R^r_{jk} = 0.$$

Also in an h-isotropic non-Riemannian F^n the following are equivalent (Proposition 4 of Hashiguchi[2])

$$\text{(a)}\quad R_{hijk} = 0,\quad \text{(b)}\quad C_{hir}\,R^{r}_{jk} = 0,\ \text{(c)}\ P_{hij|k} = P_{hik|j},$$
$$\text{(d)}\quad P_{hij|0} = 0 \qquad \ldots(3.15)$$

and $P^{m}_{hj}\,P_{mik} - j \mid k = 0$ holds (Proposition 3 of Hashiguchi[2]). From these results and the relations (1.2), (1.3) and (1.4) we have result.

Theorem 3.5—An h-isotropic is RBF^n-space if and only if it is BHF^n. Especially, if BHF^n-space is an h-isotropic space, then Theorem 2.2 holds.

PROOF : As the condition (3.1) directly follows from Theorem 2.2, thus an h-isotropic RBF^n-space turns to BHF^n-space.

Conversely, in an h-isotropic BHF^n-space, conditions (2.1a) and (2.1b) hold i.e. it turns to RBF^n-space.

Especially, if BHF^n-space is h-isotropic, then it is RBF^n-space by virtue of above Theorem, and hence Theorem 2.2 will hold good.

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