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ON NEAR-FIELDS

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In this paper six equivalent conditions are obtained for a near-ring to be a near-field.

1. INTRODUCTION

The following results are known :

(i) Let R be a ring with more than one element. Then the non-zero elements of R from a group under multiplication if and only if for every $a \neq 0$ in R, there exists a unique b in R such that $aba = a$. (see McCoy[4], p. 10).

(ii) Let R be a zerosymmetric near-ring which contains a distributive element $r \neq 0$. Then R is a near-field if and only if for each $a \neq 0$ in R, $Ra = R$. (see Ligh[2], 2.3).

(iii) Let R be a finite zerosymmetric near-ring that contains a distributive element $w \neq 0$ and for each $x \neq 0$ in R there exists $y \in R$ such that $yx \neq 0$. Then R is a near-field if and only if R has no proper left ideal. (see Ligh[2], 2.6).

In this paper we will show that (i) can be generalized to near-rings. Ligh[3] has proved the same result with the assumption that the near-ring contains a non-zero distributive element. Hence we are able to drop the requirement of a non-zero distributive element.

The zerosymmetric condition in (ii) and (iii) can be dropped and the requirement of a non-zero distributive element can be dropped by imposing the condition that idempotents commute. We give example of a near-ring which does not have any non-zero distributive element but idempotents commuting and we also show that (iii) can be extended to infinite near-ring if the near-ring is regular.

Throughout this paper we consider only right near-rings. We have given results (ii) and (iii) for right near-rings. Ligh[2] has proved the corresponding results for left near-rings. N always stands for right near-ring. N is called zerosymmetric if $n0 = 0$ for

all $n \in N$. N is called a near-field if $(N^* = N|_{\{0\}}, .)$ is a group. An element e in N is called an idempotent if $e^2 = e$. An element r in N is called distributive if $r(n_1 + n_2) = rn_1 + rn_2$ for all $n_1, n_2 \in N$. A subgroup Δ of N is called a N-subgroup if $N\Delta \subset \Delta$. A normal subgroup L of N is called a left ideal if $n(n'+1) - n\,n' \in L$ for all $n, n' \in N$ and $l \in L$. A near-ring N is called regular if for each $a \in N$ there exists $x \in N$ such that $axa = a$. For the basic terminology and notation we refer to Pilz[5]. All our near-rings will have atleast two elements.

2. Conditions to be a Near-Field

Theorem 1—Let N be a near-ring. Then N is a near-field if and only if for every $a \neq 0$ in N, there exists a unique b in N such that $aba = a$.

Proof : Let $a \neq 0$ and $b \neq 0$ in N, then $ab \neq 0$. For, if not there exists $x \in N$ such that $bxb = b$. Now $b(x - a)\ b = bxb = b$. By uniqueness $x - a = x$. Hence $a = 0$, contradiction. Hence N is without zero divisors. N is zerosymmetric, since for any $n \in N$ $(n0)\ (0)\ (n0) = n0$ and $(n0)\ (n0)\ (n0) = n0$. Hence by uniqueness $n0 = 0$. Given $a \neq 0$ in N, there exists a unique b in N such that $aba = a$. So $a(bab)\ a = (aba)\ ba = aba = a$. By uniqueness $bab = b$. Note ab, ba are non-zero idempotents.

Let e, f be any two non-zero idempotents. Then $ef \neq 0$ and there exists $x \in N$ such that $(ef)\ (x)\ (ef) = ef$ and $x\,(ef)\,x = x$. Let $y = xe$, then $(ef)\ (y)\ (ef) = ef$. Hence by uniqueness $x = y = xe$. Similarly if $y = fx$, we have $x = y = fx$. So $x^2 = (xe)\,(fx) = x\,(ef)\,x = x$. Hence $x^3 = x$ and $x\,(ef)\,x = x$. By uniqueness $x = ef$ which is an idempotent. So $(ef)\,e\,(ef) = ef$ and $(ef)\,f\,(ef) = ef$. Hence $e = f$. So N contains only one non-zero idempotent say e.

We have shown that N^* is closed. For any $a \in N^*$, there exists $b \in N$ such that $aba = a$. Since ba is a non-zero idempotent $ae = a$. Hence e is right identity for N^*. Since ab is a non-zero idempotent $ab = e$. Hence b is the right inverse of a. Hence $(N^*, .)$ is a group. So N is a near-field. The other part of the theorem is immediate.

Hence we are able to drop the requirement of a non-zero distributive element in (Ligh[3], 5.1).

Corollary 1 (Ligh[2], 3.1)—Let R be a distributively generated near-ring with more than one element. Then R is a division ring if and only if for each $a \neq 0$ in R, there exists a unique b in R such that $aba = a$.

Proof : By Theorem 1, R is a near-field. Hence by Pilz[5] $(R, +)$ is abelian and by (Pilz[5], p. 174) R is a ring. Hence R is a division ring.

Theorem 2—Let N be a near-ring in which idempotents commute. Then N is a near-field if and only if for each $a \neq 0$ in N, $Na = N$ and $N0 \neq N$.

PROOF : Let $a \neq 0$ and $b \neq 0$ be in N. Then $ab \neq 0$. If not then $Na = N$, so $Nab = Nb$. Hence $N0 = Nb = N$, a contradiction. Hence N is without zero divisors. Given $a \neq 0$ in N, there exists $y \in N$ such that $ya = a$. So $(a - ay)\, a = 0$. Hence $a = ay = ya$. There exists $x \in N$ such that $xa = y$. Now $axa = ay = a$. Hence N is regular. Let $b = xax$. Then $aba = a\ (xax)\, a = (axa)\, xa = axa = a$ and $bab = (xax)\, a\, (xax) = x\, (axa)\, xax = x\, (axa)\, x = xax = b$.

Now let us show that there exists only only one b in N satisfying $aba = a$. If possible let us assume there exists $x \in N$ satisfying $axa = a$.
As ba and xa are idempotents we have,

$$(xab - b)\, ab = (xab)\, (ab) - bab = (xa)\, (ba)\, b - bab = (ba)\, (xa)\, b - bab$$
$$= b\, (axa)\, b - bab = bab - bab = 0.$$

Since $ab \neq 0$, we have $xab = b$. So $(b - x)\, ab = bab - xab = b - b = 0$. So $b = x$. Hence by Theorem 1, N is a near-feld. The other part of the theorem is immediate as given $a \neq 0 \in N$ and for any $n \in N$, $n = n1 = na^{-1}\, a = (na^{-1})\, a$. Hence $n \in Na$. So $N \subset Na$. Hence $N = Na$ and clearly $N0 \neq N$.

The following example shows that in Theorem 2, we cannot drop the condition that idempotents commute.

Example 1—Let $(N, +)$ be any group having atleast three elements. Define multiplication on N as

$ab = a$ for all a and $b \neq 0$ in N

$= 0$ if $b = 0$.

The next example shows that there exists near-rings without non-zero distributive element but idempotents commuting.

Example 2—On the group of addition module 6 $(Z_6, +)$ define multiplication as shown in the Table below:

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	4	0
2	0	2	0	0	2	0
3	0	3	0	0	0	0
4	0	4	0	0	4	0
5	0	5	0	0	2	0

Theorem 3—Let N be a zerosymmetric near-ring in which idempotents commute and suppose that for each $x \neq 0$ in N, there exists $y \in N$ possibly depending on x such that $yx \neq 0$. Then N is a near-field if and only if N has no proper N-subgroup.

PROOF : Suppose N has no proper N-subgroup. For each $x \neq 0 \in N$, Nx is a N-subgroup of N. Since there exists $y \in N$ such that $yx \neq 0$, $Nx \neq 0$. Hence $Nx = N$ and clearly $N0 \neq N$. Hence by Theorem 2, N is a near-field. The other part of the theorem is immediate, since let $\Delta \neq 0$ be a N-subgroup and let $a \neq 0 \in \Delta$. Then $a^{-1} a = 1 \in \Delta$. So for any $n \in N$, $n = n\,1 \in \Delta$. So $N = \Delta$.

Theorem 4—Let N be a finite near-ring in which idempotents commute and suppose that for each $x \in N$ there exists $y \in N$ such that $yx \neq 0$. Then N is a near-field if and only if N has no proper left ideal and $N0 \neq N$.

PROOF : Suppose N has no proper left ideal. Let $x \neq 0$ in N and let $K(x) = \{n \in N: nx = 0\}$. Clearly $K(x)$ is a left ideal. Since there exists $y \in N$ such that $yx \neq 0$, we have $y \notin K(x)$. So $K(x) = 0$. Let $\phi : (N, +) \to (Nx, +)$ given by $\phi(n) = nx$. Then ϕ is an isomorphism. Since N is finite, $Nx = N$. Hence by Theorem 2, N is a near-field. The other part is trivial since N is a near-field, it is zerosymmetric and hence every left ideal is a N-subgroup.

Theorem 5—Let N be a near-ring that contains a distributive element $r \neq 0$ and suppose that for each $x \neq 0 \in N$, there exists $y \in N$ such that $yx \neq 0$. Then N is a near-field if and only if N is regular and N has no proper left ideal.

PROOF : Suppose N is regular with no proper left ideal. Let $b \neq 0 \in N$ and let $K(b) = \{n \in N : nb = 0\}$. Then $K(b)$ is a left ideal. Since there exists some $t \in N$ such that $tb \neq 0$, we have $K(b) = 0$. So if $a \neq 0 \in N$ and $b \neq 0 \in N$, then $ab \neq 0$. Hence N is without zero divisors.

Since N is regular, there exists $x \in N$ such that $rxr = r$. Let $rx = e$. So $er = r$ and $e \neq 0$ and $e^2 = e$.

So $(re - r)e = 0$. Hence $r = re = er$. Let $z \in N$. Then $(ze - z)r = 0$. So $ze = z$. Moreover $r(ez - z) = 0$. Hence $z = ez = ze$. So e is identity in N. If $K(0) = \{n \in N : n0 = 0\}$, then $K(0)$ is a left ideal and since $e \in K(0)$, $K(0) \neq 0$. So $K(0) = N$ and N is zerosymmetric. Hence by Beidleman[1], N is a near-field. The other part is immediate.

Theorem 6—Let N be a near-ring in which idempotents commute. N is a near-field if and only if N is regular and N has no proper left ideal.

PROOF : Suppose N is regular with no proper left ideal. For any $n \in N$, $(n0)^2 = n0(n0) = n0$. Since 0 is also idempotent, we have $0 = 0(n0) = (n0)0 = n0$.

Hence N is zerosymmetric. Since N is regular given $a \in N$ there exists $x \in N$ such that $axa = a$. Let $b = xax$, then $aba = a$ and $bab = b$. So ab and ba are idempotents. For any $a \in N$ if $a^2 = 0$, we have there exists $b \in N$ such that $aba = a$ and $bab = a$. So $b^2 = (bab)(bab) = b(ab)(ba)b = b(ba)(ab)b = b^2a^2b^2 = 0$ and $[a(ba + b)]^2 = a(ba + b)a(ba + b) = [a(ba + b)a](ba + b) = aba(ba + b) = a(ba + b)$.

Hence $0 = ba^2(ba + b) = (ba)[a(ba + b)] = a(ba + b)ba = ababa = aba = a$.

Hence if $a \neq 0 \in N$, then $a^2 \neq 0$.

Let $b \neq 0 \in N$. Let $K(b) = \{n \in N : nb = 0\}$. Since $b \notin K(b)$, we have $K(b) = 0$. So if $a \neq 0$ and $b \neq 0 \in N$, then $ab \neq 0$ and N is without zero divisors. Let e be any non-zero idempotent. Then for any $n \in N$, we have $(ne - n)e = 0$ and hence $ne = n$. Hence every non-zero idempotent is right identity. Let e, f be any two non-zero idempotents in N, then $e = ef = fe = f$. So N contains a unique non-zero idempotent say e. Let a be any element in N^*, then there exists $b \in N$ such that $aba = a$. Since ba is a non-zero idempotent, $ba = e$ and $ae = a$. Hence e is right identity for N^*. Since ab is a non-zero idempotent $ab = e$. So b is the right inverse of a. Hence $(N^*,.)$ is a group. So N is a near-field. The other part is immediate.

References

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