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THE f-PRIME RADICAL IN NEAR-RINGS

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The notion, 'f-strongly nilpotent element' is introduced and a characterization of the f-prime radical of a near-ring in terms of f-strongly nilpotent elements is obtained. As a consequence of this, we obtain a necessary and sufficient condition for the prime radical to be equal to the f-prime radical.

1. INTRODUCTION

Van der Walt[8] defined s-prime ideals in non-commutative rings and obtained analogous results of McCoy[3] for s-prime ideals. Again Van der Walt[9] carried out some of his earlier[8] results of to near-rings. Murata *et al.*[4] defined f-prime ideals and achieved some results similar to those by Van der Walt. Hsu[2] and Satyanaryana[7] have extended some of these results to Γ-rings.

Sambasiva Rao and Satyanarayana[6] obtained a characterization of the prime radical in near-rings in terms of strongly nilpotent elements and Groenewald and Potgieter[1] have extended some results of Hsu[2] and Satyanarayana[7] to f-prime and f-semiprime ideals in near-rings.

Throughout by a near-ring N we mean a zero-symmetric near-ring with right distributive property. We shall utilize the standard notations and definitions given in Pilz[5]. For any subset X of N the set $\{x \in N | x \notin X\}$ is denoted by $C(X)$. The ideal generated by an element $x \in N$ is denoted by (x). Unless otherwise specified '0' stands for the additive identity in N.

Henceforth f stands for a mapping from N into the set of all ideals of N satisfying the following conditions

(i) $a \in f(a)$

(ii) $x \in f(a) + A \Rightarrow f(x) \subseteq f(a) + A$ for any ideal A of N.

Definitions—A subset S of N is an f-system if it contains an m-system S^*, called the kernel of S such that of $S = \phi$ or $f(s) \cap S^* \neq \phi$ for every $s \in S$. An ideal P of N is said to be f-prime if $C(P)$ is an f-system. The prime radical, $r(N)$, of N is defined as the intersection of all prime ideals of N. The f-prime radical, $r_f(N)$, of N is defined as the intersection of all f-prime ideals of N, equivalently $r_f(N)$ is the set of all those elements $x \in N$ for which every f-system that contains x must contains 0. We say that a sequence $x_1, x_2, \ldots$ of elements of N 'Vanishes (or terminates) after a certain stage' if there exists a positive integer k such that $x_s = 0$ for all $s \geqslant k$. An element x in N is said to be strongly nilpotent if every sequence $x_1, x_2 \ldots$ of elements of N with $x_1 = \bar{x}$ and $x_t = x_{t-1}\, x_{t-1}^*$ for some $x_{t-1}^* \in (x_{t-1})$ must terminate after a certain stage. An element a in N is said to be f-strongly nilpotent if every element of $f(a)$ is strongly nilpotent.

2. The f-Prime Radical

We start this section with the following theorem.

Theorem 2.1—Let N be a near-ring in which $0 \neq r(N) \neq N$ and $x \in C(r(N))$. Define $f(a) = (\{a, x\})$ for every $a \in N$. Then N contains no f-strongly nilpotent elements. In particular the element '0' is not strongly nilpotent.

Proof : In a contrary way suppose there exists an f-strongly nilpotent element y in N. Then every element of $f(y)$ is strongly nilpotent and so by Theorem 3 of Sambasiva Rao and Satyanarayana[6], $f(y) \subseteq r(N)$, which implies $x \in r(N)$, a contradiction. Thus N contains no f-strongly nilpotent elements.

In general a strongly nilpotent element may not be f-strongly nilpotent.

Example 2.1—Let $X = \{0, b\}$ and define $+$ and . on X as

$$0 + 0 = 0 + b = b + b = b + 0 = 0$$

$$0.b = b.b = b.0 = 0.0 = 0.$$

Now $(X, +, .)$ is a near-ring. Write $N = Y \oplus X$, the direct sum of near-rings Y and X where $Y = Z_6$, the ring of integers modulo 6. The fact $(\{0\} \oplus X)^2 = \{0\}$ implies (0) is not a semiprime ideal of N and since $r(N)$ is semiprime, we have $(0) \neq r(N)$. Secondly, it can be easily verify that the ideal $\{0, 2, 4\} + X$ of N is a prime ideal of N and so $r(N) \subseteq \{0, 2, 4\} \oplus X \subseteq N$. Thus $(0) \neq r(N) \neq N$. Let $x \in C(r(N))$ and define $f(a) = (\{a, x\})$ for all $a \in N$. Let $0 \neq y \in r(N)$. Then by Theorem 2.1; y is not f-strongly nilpotent, but by Theorem 3 of Sambasiva Rao and Satyanarayana[6], y is strongly nilpotent.

Lemma 2.1—Let $a_1, a_2, \ldots$ be a sequence of elements of N with $a_i = a_{i-1}\, a^*_{i-1}$ for some $a^*_{i-1} \in (a_{i-1})$ for all $i \geqslant 2$. Then $\{a_i / i \geqslant 1\}$ is an m-system.

PROOF : Straight forward verification.

Theorem 2.2—Let N be a near-ring. Then

$$r_f(N) = \{ x \in N / x \text{ is } f\text{-strongly nilpotent}\}.$$

PROOF : Let $0 \neq x \in r_f(N)$. Suppose x is not strongly nilpotent, that is there is a sequence $a_1, a_2, \ldots$ of elements of N such that $a_1 = y \in f(x)$ and $a_i = a_{i-1}\, a^*_{i-1}$ for some $a^*_{i-1} \in (a_{i-1})$ and $a_{i-1} \neq 0$ for all $i \geqslant 2$. By Lemma 2.1; $S = \{a_i / i \geqslant 1\}$ is an m-system. Now S is an f-system with kernel $S^* = S \setminus \{x\}$ and $S \cap \{0\} = \phi$. By Lemma 1.5 (a) of Groenewald and Potgieter[1], there exists an f-prime ideal P such that $P \cap S = \phi$. Now $x \notin P$ and so $x \notin r_f(N)$, a contradiction. Thus x is f-strongly nilpotent.

Conversely suppose that x is an f-strongly nilpotent element. If $x \notin r_f(N)$, then there exists an f-prime ideal P such that $x \notin P$. Now $K = C(P)$ is an f-system. Let K^* be a kernel of the f-system K. Since $P \cap K^* = \phi$ and K^* is an m-system by Theorem 2 of Van der Walt[9] there exists a prime ideal P^* such that $P \subseteq P^*$ and $P^* \cap K^* = \phi$. Since $x \in K$, there exists an element y in $f(x) \cap K^*$. So $y \in K^* \subseteq C(P^*) \subseteq C(r(N))$ which implies $y \notin r(N)$. Now $y \in f(x)$ and by Theorem 3 of Sambasiva Rao and Satyanarayana[6], y is not strongly nilpotent. This is a contradiction to the fact x is f-strongly nilpotent. This completes the proof of the theorem.

Theorem 2.3—(I) Let P be an ideal of N such that $f(0) \subseteq P$. Then P is prime if and only if P is f-prime.

(II) If $f(0) \subseteq r_f(N)$, then

(a). An ideal P is prime if and only if P is f-prime.

(b). $r_f(N) = r(N)$.

(III) (a). If $f(0) = (0)$, then $r(N) = r_f(N)$.

(b). Suppose $f(0) \neq (0)$. Then $r(N) = r_f(N) \neq (0)$ if and only if $f(0) \subseteq r_f(N)$.

PROOF : (I) It is trivial that every prime ideal is f-prime. To show the converse part suppose P is f-prime. It is suffices to show $(a)(b) \subseteq P$, $a, b \in N$ implies either $a \in P$ or $b \in P$. Suppose $(a)(b) \subseteq P$. Now by definition of (f)

$$a \in f(0) + (a) \text{ implies } f(a) \subseteq f(0) + (a)$$

and

$$b \in f(0) + (b) \text{ implies } f(b) \subseteq f(0) + (b).$$

Now $f(a) f(b) \subseteq ((a) + f(0))((B) + f(0)) \subseteq (a)(b) + f(0) \subseteq P$.

Since P is f-prime by Lemma 1.2 (a) of Groenewald and Potgieter[1], either $a \in P$ or $b \in P$.

II (a). If P is f-prime, then P contains $r_f(N)$ and so $P \supseteq f(0)$. Now it follows from part I.

II (b) : follows from II (a).

II (a) : If $f(0) = (0)$, then $f(a) = (a)$ for all $a \in N$ and so the two concepts f-prime and prime are same. Thus $r_f(N) = r(N)$.

III (b). Let $f(0) \neq (0)$. Suppose $r(N) = r_f(N) \neq 0$.

Let $0 \neq x \in r_f(N)$. By Theorem 2.2; x is f-strongly nilpotent. Now by Theorem 3 of Sambasiva Rao and Satyanarayana[6], $f(x) \subseteq r(N)$. Since $0 \in f(x) + (0)$, $f(0) \subseteq f(x) + (0) = f(x)$ and hence $f(0) \subseteq f(x) \subseteq r(N) \subseteq r_f(N)$. The converse part follows from II(b).

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