Indian J. pure appl. Math., **17** (3) : 331-337 March 1986

PRIME α-IDEALS IN A *O*-DISTRIBUTIVE LATTICE

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(*Received* 8 *October* 1984; *after revision* 22 *July* 1985)

A new characterization of a quasi complemented lattice is established and using this, some equivalent conditions are obtained for $\mathcal{MM}(L)$ (the space of minimal prime ideals of *O*-distributive lattice L with hull-kernel topology) to be a compact and discrete space

1. INTRODUCTION

In this paper, we introduce a topology on the set $\nu(L)$ of all prime α-ideals (in the sense of Cornish[1]) of a *O*-distributive (in the sense of Varlet[5]) lattice L. It is proved that this topology is a T_0-space with the family of compact open subsets of $\nu(L)$ as a base for the open sets. Some equivalent conditions are given for every prime α-ideal of L to be a minimal prime ideal. Also it is shown that L is quasicomplemented if and only if the exterior of every compact open subset of $\nu(L)$ is compact. Further some necessary and sufficient conditions are established for $\mathcal{MM}(L)$ (the space of minimal prime ideals of L with hull-kernel topology) to be a compact and discrete space.

2. PRELIMINARIES

For any ideal I in a lattice $L = (L; \vee, \wedge, 0)$ with 0, we denote $I^* = \{x \in L \mid x \wedge y = 0 \text{ for all } y \in I\}$ and $I' = \{x \in L \mid x \in (a)^{**} \text{ for some } a \in I\}$. ($I$ is said to be non dense if $I^* \neq \{0\}$. I is called an α-ideal (in the sense of Cornish[1]) if $x \in I$ implies $(x)^{**} \subseteq I$. The lattice of all ideals of L is denoted by $I(L)$ (lattice operations in $I(L)$ are denoted by $\vee$ and $\cap$ respectively).

According to Varlet[5], a lattice L with 0 is called quasicomplemented if for any $a \in L$ there is $b \in L$ such that $a \wedge b = 0$ and $(a \vee b)^* = (0)$. A lattice L with 0 is *O*-distributive if for any $a, b, c \in L$, $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. In Varlet[5] it is proved that a *O*-distributive lattice L is quasicomplemented if and only if for any $x \in L$, $(x)^{**} = (y)^*$ for some $y \in L$. Also it is not hard to show that in a *O*-distributive lattice L, for ony ideal I in L, I' is the smallest α-ideal containing I. A lattice L with 0 and 1 is called semicomplemented if for any $a \in L$ ($a \neq 1$) there is $b \neq 0 \in L$ such that $a \wedge b = 0$.

For the rest of this paper, L denotes a O-distributive lattice. The principal ideal (filter) generated by an element $a \in L$ is denoted by $(a]$ ($[a)$). An element $a \in L$ is said to be dense if $(a)^* = \{0\}$. All ideals (filters) are assumed to be proper. Most of the topological concepts used in this paper are found in Kelley[4] and Venkatanarasimhan[6,7].

3. A Topology for the Prime α-Ideals

Let $\nu(L) = \{P \mid P$ is a prime α-ideal of $L\}$ and for each $x \in L$, let $\nu(x) = \{P \in \nu(L) \mid x \notin P\}$. Then $\nu(L)$ can be given a topology that is compact if and only if L has a dense element.

Theorem 1—The class $\mu_L = \{\nu(x) \mid x \in L\}$ forms a base for the open sets for a topology on $\nu(L)$, and further $\nu(L)$ is a T_0-space (with this topology) and the set of all compact open subsets of $\nu(L)$ is μ_L. Also $\nu(L)$ is compact if and only if L has a dense element.

To prove this theorem, we need the following lemma.

Lemma 1 (Jayaram[3], Lemma 1)—Let I be an α-ideal of L and S be a meet subsemilattice of L such that $I \cap S = \phi$. Then there is a prime α-ideal P of L such that $I \subseteq P$ and $P \cap S = \phi$.

Proof of Theorem 1—Clearly $\nu(x) \cap \nu(y) = \nu(x \wedge y)$ and $\bigcup_{\nu(x) \in \mu_L} \nu(x) = \nu(L)$, and so μ_L form a base for the open sets for a topology on $\nu(L)$. We first show that each $\nu(x)$ is compact in $\nu(L)$. Let $\nu(x) \subseteq \bigcup_{\alpha \in \Delta} \nu(x_\alpha)$ and let I be the ideal generated by $\{x_\alpha \mid \alpha \in \Delta\}$. We claim that $x \in I'$. If $x \notin I'$, then $I' \cap [x) = \phi$ and so by Lemma 1, there is a prime α-ideal P such that $I' \subseteq P$ and $x \notin P$; so that $P \in \nu(x) \subseteq \bigcup_{\alpha \in \Delta} \nu(x_\alpha)$ and hence $x_\alpha \notin P$ for some $\alpha \in \Delta$ which contradicts the fact that $I \subseteq I' \subseteq P$. Therefore $x \in I'$ and so $x \in (y)^{**}$ for some $y \in I$. Since I is the ideal generated by $\{x_\alpha \mid \alpha \in \Delta\}$, it follows that $y \leqslant x_{\alpha_1} \ldots x_{\alpha_n}$ for some $x_{\alpha_i} \in \{x_\alpha \mid \alpha \in \Delta\}$; so that $x \in (y)^{**} \subseteq (x_{\alpha_1} \vee \ldots \vee x_{\alpha_n})^{**}$. Now it can be easily seen that $\nu(x) = \bigcup_{i=1}^{n} \nu(x_{\alpha_i})$ which is a finite subcover and therefore $\nu(x)$ is compact. Thus for each $x \in L$, $\nu(x)$ is a compact open subset of $\nu(L)$.

Let X be a compact open subset of $\nu(L)$. Since X is open, $X = \bigcup_{\alpha \in \Delta} \nu(x_\alpha)$. As X is compact, it follows that $X = \bigcup_{i=1}^{n} \nu(x_{\alpha_i})$ for some $\alpha_1, \ldots, \alpha_n \in \Delta$. So $X = \nu(x_{\alpha_1} \vee \ldots \vee x_{\alpha_n})$. Thus μ_L is the setof all compact open subsets of $\nu(L)$.

Next we prove that $\nu(L)$ is a T_0-space. Suppose $P \neq Q$ for some $P, Q \in \nu(L)$. Then either $P \subsetneq Q$ or $Q \not\subseteq P$. Without loss of generality we assume that $P \not\subseteq Q$.

Then there is an element $x \in P$ such that $x \notin Q$. So $P \notin \nu(x)$ and $Q \in \nu(x)$. Therefore $\nu(L)$ is a T_0-space.

Now we prove that $\nu(L)$ is compact if and only if L has a dense element. Suppose L has a dense element say d. Then $d \notin P$ for every prime α-ideal of L. Consequently $\nu(d) = \nu(L)$ and hence $\nu(L)$ is a compact space. Conversely assume that $\nu(L)$ is compact. Then $\nu(L) = \nu(d)$ for some element $d \in L$. It is enough if we show that d is a dense element. i. e., $(d)^* = (0)$. Observe that d does not belong to any prime α-ideal of L. Suppose $(d)^* \neq (0)$. Then there is an element $x \in L$ such that $x \in (d)^*$ and $x \neq 0$. Since $x \in (d)^*$ and $x \neq 0$, it follows that $x \notin (d)^{**}$. So $(d)^{**} \cap [x] = \phi$ and therefore by Lemma 1, there is a prime α-ideal P such that $(d)^{**} \subseteq P$ and $x \notin P$. Thus there is a prime α-ideal P of L such that $d \in P$ which is a contradiction. Hence $(d)^* = (0)$. This completes the proof of the theorem.

For any ideal I of L, let $\nu(I) = \{P \in \nu(L) \mid I \not\subseteq P\}$. It can be easily verified that I is an α-ideal if and only if $I = \bigcup_{x \in I} \nu(x) = \bigcup_{x \in I} \nu((x)^{**})$. Therefore every open subset of $\nu(L)$ is of the form $\nu(I)$ for some α-ideal I of L.

Let us denote the set of all α-ideals of L by $\alpha(L)$. Then it can be easily seen that $\alpha(L) = (\alpha(L); \textcircled{\vee}, \cap)$ is a complete distributive lattice with (0] as the least element, L as the greatest element and set theoretic intersection as the infimum. The supremum of any $I, J \in \alpha(L)$ is defined by $I \textcircled{\vee} J = (I \vee J)'$. In fact we show that $\alpha(L)$ is isomorphic to the lattice of all open subsets of $\nu(L)$.

Theorem 2—The lattice $\alpha(L) = (\alpha(L); \textcircled{\vee}, \cap)$ is isomorphic to the lattice of all open subsets of $\nu(L)$ and the mapping $I \to \nu(I)$ $(I \in \alpha(L))$ takes arbitrary lattice sums to corresponding set unions.

PROOF : We observed above that every open subset of $\nu(L)$ is of the form $\nu(I)$ for some $I \in \alpha(L)$, so that the map $I \to \nu(I)$ from $\alpha(L)$ to $\mathcal{L}$, the lattice of all open subsets of $\nu(L)$ is onto. Also from Lemma 1, the above map is one-one. Again for $\{I_\alpha\} \subseteq \alpha(L)$, $\nu(\textcircled{\vee}_{\alpha \in \Delta} I_\alpha) = \nu((\bigvee_{\alpha \in \Delta} I_\alpha)') = \bigcup_{\alpha \in \Delta} \nu(I_\alpha)$ and $\nu(I_\alpha \cap_{\alpha, \beta \in \Delta} I_\beta) = \nu(I_\alpha) \cap \nu(I_\beta)$. Therefore $\alpha(L)$ is isomorphic to the lattice of all open subsets of $\nu(L)$. This completes the proof of the theorem.

It is well known that minimal prime ideals exist in L and that they are precisely the set theoretic complements of maximal filters in L (see also Cornish[2]). Let us denote the class of all minimal prime ideals of L by $\mathcal{M}(L)$, and for each $x \in L$, let $\mathcal{M}(x) = \{P \in \mathcal{M}(L) \mid x \notin P\}$. Then the sets of the form $\mathcal{M}(x)$ $(x \in L)$ form an open base for the open sets for a Hausdorff topology on $\mathcal{M}(L)$. We now establish some equivalent conditions for every prime α-ideal to be a minimal prime ideal.

Theorem 3—The following statements are equivalent :

(i) L is sectionally quasicomplemented (i. e., each interval $[0, a] = \{x \in L \mid 0 \leqslant x \leqslant a\}$ is a quasicomplemented lattice)

(ii) For all $x, y \in L$, there is $x' \in (x)^*$ such that

$$\mathscr{M}(y) \cap (\mathscr{M}(L) - \mathscr{M}(x)) = \mathscr{M}(y \wedge x')$$

(iii) $\nu(L) = \mathscr{M}(L)$

(iv) Each $\nu(x)$ is closed in $\nu(L)$

(v) $\nu(L)$ is Hausdorff

(vi) For every $y \in L$, $\mathscr{M}(y)$ is compact as a subset of $\mathscr{M}(L)$.

PROOF : (i) $\Rightarrow$ (ii)—Suppose (i) holds. Let $x, y \in L$. Since $x \in [0, x \vee y]$, by (i) there is $x' \in [0, x \vee y]$, such that $x \wedge x' = 0$ and $x \vee x'$ is dense in $[0, x \vee y]$. Now it can be easily seen that $\mathscr{M}(y) \cap (\mathscr{M}(L) - \mathscr{M}(x)) = \mathscr{M}(y) \wedge x')$. Therefore (ii) holds.

(ii) $\Rightarrow$ (iii)—Suppose (ii) holds. Clearly $\mathscr{M}(L) \subseteq \nu(L)$. Let $P \in \nu(L)$. Choose any $x \in P$ and $y \notin P$. Then by (ii) there is $x' \in (x)^*$ such that $\mathscr{M}(y) \cap (\mathscr{M}(L) - \mathscr{M}(x)) = \mathscr{M}(y \wedge x')$, so that $(y)^* = ((y \wedge x) \vee (y \wedge x'))^*$. Consequently $x' \notin P$. Thus for every $x \in P$ there is $x' \notin P$ such that $x \wedge x' = 0$ and therefore $P \in \mathscr{M}(L)$. Hence $\mathscr{M}(L) = \nu(L)$.

(iii) $\Rightarrow$ (iv)—Suppose (iii) holds. Then for every $x \in L$, $\nu(x) = \mathscr{M}(x)$ which is closed in $\mathscr{M}(L) = \nu(L)$, and so (iv) holds.

(iv) $\Rightarrow$ (v)—Suppose (iv) holds. Let P and Q be two distinct prime α-ideals of L. Choose $x \in P$ such that $x \notin Q$. Then $Q \in \nu(x)$ and $P \in \nu(L) - \nu(x)$. Since $\nu(x)$ is closed, there is $y \in L$ such that $P \in \nu(y) \subseteq \nu(L) - \nu(x)$. Also $\nu(x) \cap \nu(y) = \phi$. Thus (v) holds.

(v) $\Rightarrow$ (vi)—Suppose (v) holds. Then $\nu(L) = \mathscr{M}(L)$. Now the result follows from Theorem 1.

(vi) $\Rightarrow$ (i)—Suppose (vi) holds. Let $x \in [0, y]$ $(x, y \in L)$.

Clearly $\mathscr{M}(y) \subseteq \mathscr{M}(x) \cup \{\cup \mathscr{M}(t) : t \in (x)^*\}$. Since $\mathscr{M}(y)$ is compact, there exists $t_1, \ldots, t_n \in (x)^*$ such that

$$\mathscr{M}(y) \subseteq \mathscr{M}(x) \cup \{\bigcup_{i=1}^{n} \mathscr{M}(t_i) : t_i \in (x)^*\}. \text{ Put } x' = \bigvee_{i=1}^{n} t_i. \text{ Then } \mathscr{M}(y)$$

$\subseteq \mathscr{M}(x) \cup \mathscr{M}(x')$. Now it can be easily seen that $x \wedge (x' \wedge y) = 0$ and $x \vee (x' \wedge y)$ is dense in $[0, y]$. Therefore (i) holds. This completes the proof of the theorem.

Remark : It can be easily shown that L is quasicomplemented if and only if L is sectionally quasicomplemented and contains a dense element. Therefore from Theorems 1 and 3, it follows that L is quasicomplemented if and only if $\mathcal{M}(L)$ is a Boolean space. (A topological space X is called a Boolean space if X is compact, Hausdorff and has a base having compact open sets) (see also Cornish[2], Proposition 1.2).

Let A be a subset of a topological space T. The interior of the complement of A is called the exterior of A. According to Venkatanarasimhan[6] an open subset of a topological space T is said to be an open domain if it is identical with the interior of its closure. An open subset of T is called semi regular if it is an union of open domains. T is called semiregular if every open subset of T is semiregular.

We shall denote the closure, interior and exterior of a subset X of $\nu(L)$ by cl.X, int. X and ext. X respectively. Clearly for any subset X of $\nu(L)$, cl. $X = \{P \in \nu(L) \mid \bigcap_{q \in X} q \subseteq P\}$. Also it is not hard to show that for any ideal $I \in \mathfrak{a}(L)$ (i) cl. $\nu(I) = \nu(L) - \nu(I^*)$, (ii) int. $(\nu(L) - \nu(I)) = \nu(I^*)$ and (iii) $\nu(I)$ is an open domain if and only if $I = I^{**}$. As a consequence it can be easily seen that $\nu(L)$ is semiregular. Using these facts we shall establish a necessary and sufficient condition for L to be a quasicomplemented lattice.

Theorem 4—L is quasicomplemented if and only if the exterior of every compact open subset of $\nu(L)$ is compact.

PROOF : Suppose L is quasicomplemented. Let X be a compact open subset of (L). Then by Theorem 1, $X = \nu(x)$ for some $x \in L$. Now ext. X = ext. $\nu(x)$ = ext. $\nu((x)^{**}) = \nu((x)^*)$. As L is quasicomplemented, it follows that $(x)^* = (y)^{**}$ for some $y \in L$. Therefore $\nu((x)^*) = \nu((y)^{**}) = \nu(y)$ which is compact. Therefore the exterior of every compact open subset of $\nu(L)$ is compact.

Conversely, assume that the exterior of every compact open subset of $\nu(L)$ is compact. Let $x \in L$. Then $\nu(x)$ is a compact open subset of $\nu(L)$ and so ext. $\nu(x) = \nu((x^*))$ is compact and also open. Consequently by theorem 1, $\nu((x)^*) = \nu(y) = \nu((y)^{**})$ for some $y \in L$; so that by Theorem 2, $(x)^* = (y)^{**}$. Thus for any $x \in L$, there is $y \in L$ such that $(x)^* = (y)^{**}$. This shows that L is quasicomplemented, and completes the proof of the theorem.

In Cornish[2], it is proved that $\mathcal{M}(L)$ is compact, Hausdroff, and extremally disconnected if and only if for any ideal A in L there L is $a \in L$ such that $A^* = (a]^*$ (Theorem 2.3). The following Theorem gives another equivalent condition for $\mathcal{M}(L)$ to be compact and extermally disconneted.

Theorem 5—The following two statements are equivalecnt :

(i) For any ideal A of L, $A^{**} = (a]^*$ for some $a \in L$.

(ii) (*a*) The exterior of every compact open subset of $\nu(L)$ is compact and (*b*) the interior of the intersection of any family of compact open subsets of $\nu(L)$ is compact.

PROOF : Suppose (*i*) holds. Then L is quasicomplemented (Cornish,[2] Proposition 1.2) and so by Theorem 4, the exterior of every compact open subset of $\nu(L)$ is compact and therefore (*a*) holds.

Now we claim that (*b*) holds. Let $\{g_i \mid i \in \Delta\}$ be a family of compact open subsets of $\nu(L)$. Then for each $i \in \Delta$, there is $x_i \in \Delta$ such that $g_i = \nu(x_i)$. Since L is quasicomplemented, it follows that $\nu(L) = \mathscr{M}(L)$ and $\nu(x_i) = \mathscr{M}(x_i)$ for every $i \in \Delta$. Also for every $x_i \in L$ there is $y_i \in L$ such that $(x_i)^* = (y_i)^{**}$. Observe that int. $(\bigcap_{i\in\Delta} g_i)$ = int. $(\bigcap_{i\in\Delta} \mathscr{M}(x_i))$ = int.$(\mathscr{M}(L) - \bigcup_{i\in\Delta} \mathscr{M}(y_i))$. Let I be the ideal generated by $\{y_i \mid i \in \Delta\}$. Then

int. $(\mathscr{M}(L) - \bigcup_{i\in\Delta} \mathscr{M}(y_i))$ = int. $\mathscr{M}(L) - \mathscr{M}(I)) = \mathscr{M}(I)^*) = \mathscr{M}((a)^{**})$

(by hypothesis) $= \mathscr{M}(a)$ which is a compact open subset of $\nu(L)$. Therefore the interior of the intersection of any family of compact open subsets of $\nu(L)$ is compact.

(ii) $\Rightarrow$ (i)—Suppose (ii) holds. Then by (*a*) and Theorem 4, L is quasicomplemented and so $\nu(L) = \mathscr{M}(L)$. Let I be an ideal of L. Observe that $\mathscr{M}(I^{**})$ is an open domain and so $\mathscr{M}(I^{**})$=int. cl. $\mathscr{M}(I^{**})$ Again notice that cl. $\mathscr{M}(I^{**})$= $\mathscr{M}(L) - \mathscr{M}(I^*) = \mathscr{M}(L) - \mathscr{M}(\bigcup_{x_\alpha\in I^*} (x_\alpha)^{**}) = \bigcap_{x_\alpha\in I^*} \{\mathscr{M}(L) - \mathscr{M}((x_\alpha)^{**})\}$. Since L is quasicomplemented for each $x_\alpha \in I^*$, there is $y_\alpha \in L$ such that $(x_\alpha)^* = (y_\alpha)^{**}$. Therefore $\bigcap_{x_\alpha\in I^*} \{\mathscr{M}(L) - \mathscr{M}((x_\alpha)^{**})\} = \bigcap_{\alpha} \{\mathscr{M}((y_\alpha{}^{**})\}$. Thus $\mathscr{M}(I^{**})$ = int. $(\bigcap_{\alpha} \{\mathscr{M}((y_\alpha)^{**})\})$ and each $\mathscr{M}((y_\alpha)^{**})$ is a compact open subset of $\mathscr{M}(L)$ and so by (*b*) $\mathscr{M}(I^{**})$ is a compact open subset of $\mathscr{M}(L)$. Consequently there is $a \in L$ such that $\mathscr{M}(I^{**}) = \mathscr{M}(a) = \mathscr{M}((a)^{**})$ and hence $I^{**} = (a)^{**}$. This completes the proof the theorem.

An ideal I of L is said to be dense if $I^* = \{0\}$. An element $a \in L$ is said to be dense if $(a)^* = \{0\}$. The set of all dense elements in L is denoted by D_L.

Now we state an interesting lemma that we need.

Lemma 2—Let $\nu_0(L) = \{P \in \nu(L) \mid P = P^{**}\}$. Then the subspace $\nu_0(L)$ is discrete.

PROOF : Let $X = \{P_i \mid i \in \Delta\}$ be any subset of $\nu_0(L)$. Then for each $i \in \Delta$, $P_i = (x_i)^*$ for some $0 \neq x_i \in P_i{}^*$. Suppose $P \in$ cl. X. Then $\bigcap_{i\in\Delta} (x_i)^* \subseteq P$. If $x_i \in P$ for all $i \in \Delta$, then $P^* \subseteq \bigcap_{i\in\Delta} (x_i)^* \subseteq P$ and so $P^* = (0]$ which is absurd. So $x_i \notin P$ for some $i \in \Delta$. Consequently $P = (x_i)^*$ for some $i \in \Delta$ and hence $P \in X$. That is, $X =$ cl. X and therefore $\nu_0(L)$ is discrete.

Our final theorem establishes some equivalent conditions for $\mathscr{M}(L)$ to be a compact and discrete space.

Theorem 6—Let $D_L \neq \phi$. Then the following statements are equivalent :

(i) L is quasicomplemented and every minimal prime ideal is nondense;

(ii) $\mathscr{M}(L)$ is compact and discrete;

(iii) $\alpha(L)$ is a finite Boolean algebra;

(iv) $\alpha(L)$ is semicomplemented.

PROOF : (i) $\Rightarrow$ (ii) follows from Theorem 3 and Lemma 2.

(ii) $\Rightarrow$ (iii)—Suppose (ii) holds. Then by (ii) $\mathscr{M}(L) = \nu(L)$. Since $\nu(L)$ is compact and discrete, the lattice of all open subsets of $\nu(L)$ form a finite Boolean algebra and therefore by Theorem 2, $\alpha(L)$ is a finite Boolean algebra.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i)—Suppose (iv) holds. Clearly every minimal prime ideal of L is nondense. Let $x \in L$. Then $(x)^* \vee (x)^{**} = L$ and so $((x)^* \vee (x)^{**} \cap D_L \neq \phi$. Choose $d \in ((x)^* \vee (x)^{**}) \cap D_L$. So that $d \in (a \vee b)^{**}$ for some $a \in (x)^*$ and $b \in (x)^{**}$. Now it can be easily seen that $(x)^{**} = (a)^*$ and therefore L is quasicomplemented. This completes the proof of the theorem.

ACKNOWLEDGEMENT

Thanks are due to Professor Iqbalunnisa for her help during the preparation of this paper.

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