A BASIC HYPERGEOMETRIC TRANSFORMATION OF RAMANUJAN
AND A GENERALIZATION

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We prove in an elementary and self-contained manner a basic hypergeometric
identity of Ramanujan and its equivalence to a more general identity.

1. INTRODUCTION

The transformation

\[
(-bq)_{\infty} \sum_{n=0}^{\infty} \frac{q^n \lambda^n}{(q)_n (-bq)_n} = \sum_{n=0}^{\infty} \frac{n(n+1)}{q} \left( -\frac{\lambda}{b} \right)_n b^n
\]

...(1)

is found in the Second Notebook of Srinivasa Ramanujan [vol. II, p. 194, Entry 9]. Here,

\[
(a)_{\infty} = (a, q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)
\]

and

\[
(a)_k = (a, q)_k = \begin{cases} 
(a)_{\infty} & , k: \text{any integer}, \\
(a q^2)_{\infty} & \\
= \prod_{n=0}^{k-1} (1 - aq^n) & , k: \text{positive integer}, 
\end{cases}
\]

where \(a\) and \(q\) are complex numbers with \(|q| < 1\). A particular case \((b = 1, 
\lambda = -a)\) of (1) was first posed as an Advanced problem by L. Carlitz who also

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proved the general case by employing Euler's expansion for \((a)_r\) as a polynomial in \(a\). V. Ramamani\(^5\) has given a proof of (1) by obtaining two functional relations for the right hand side. Andrews\(^2\) has shown that (1) is a limiting case of an identity of Rogers. Ramamani and Venkatachaliengar\(^8\) have observed that (1) can be obtained as a limiting case of an identity of Heine. More recently Adiga \textit{et al.}\(^1\) have obtained two other proofs of (1). One of their proofs consists in making iterated use of Heine's identity and the other in obtaining a functional relationship for the left side of (1).

In this note we consider a more general hypergeometric series

\[
\sum_{n=0}^{\infty} \frac{n(n+1)}{q} \frac{\left( -\frac{\lambda}{a} \right)^n}{(q)_n (-bq)_n} a^n
\]

and give a simple, self-contained approach not only to proving (1) but also to show that (1) implies a more general identity

\[
(-bq) \sum_{n=0}^{\infty} \frac{n(n+1)}{q} \frac{\left( -\frac{\lambda}{a} \right)^n}{(q)_n (-bq)_n} = (-aq) \sum_{n=0}^{\infty} \frac{n(n+1)}{q} \frac{\left( -\frac{\lambda}{b} \right)^n}{(q)_n (-aq)_n} b^n
\]

or, what is the same

\[
g(a, \lambda, b, q) = g(b, \lambda, a, q).
\]

The function \(g(a, \lambda, b, q)\) in (2) was also considered by Bhargava and Adiga\(^3\) while giving a simple, unified approach to proving several continued fraction expansions of Ramanujan and Hirschhorn.

2. \textbf{Proof of the Identity (3)}

We need the following Lemma and its corollary the proofs of which are simple and have been given by Bhargava and Adiga\(^3\) earlier.

\textit{Lemma 1—If } \lvert q \rvert < 1, \text{ then } g \text{ satisfies the following functional relations :}

\[
g(a, \lambda, b, q) - g(aq, \lambda, b, q) = aq g(aq, \lambda q, bq, q), \quad \ldots(4)
\]

\[
g(a, \lambda, b, q) - g(a, \lambda q, b, q) = \lambda q g(aq, \lambda q^2, bq, q), \quad \ldots(5)
\]

\[
g(a, \lambda, b, q) - g(a, \lambda, bq, q) = bq g(aq, \lambda q, bq, q). \quad \ldots(6)
\]
Corollary 1—\( g(aq, \lambda, b, q) = (1 - aq + bq) + g(aq, \lambda q, bq, q) \\
+ (aq + \lambda q) g(aq, \lambda q^2, bq^2, q). \) ... (7)

Theorem 1—If \(|q| < 1\), then identity (1) holds.

**Proof**: Setting

\[ g(0, bt, b, q) = \sum_{n=0}^{\infty} \beta_n(t, q) b^n \] ... (8)

and putting \( a = 0, \lambda = bt \) in (7) comparing coefficients we have

\[ (1 - q^n) \beta_n = q^n (1 + iq^{n-1}) \beta_{n-1}, \quad n = 1, 2, \ldots \]

Iterating this and noting that \( \beta_n = g(0, 0, 0, q) = 1 \), we get

\[ \beta_n = \frac{n(n+1)}{q^n (-t)^n} \frac{\frac{n(n+1)}{q^n (-b)}}{(q)_n} \]

Substituting this in (8) we get (1).

Theorem 2—If \(|q| < 1\), then identity (3) holds.

**Proof**: Making use of the functional relations (4)-(6) we show that (1) implies (3). Substituting

\[ g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \beta_n(a, \lambda) b^n \]

and

\[ g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \alpha_n(\lambda, b) a^n \]

in (6) and (4) respectively and comparing coefficients we have

\[ \beta_n(a, \lambda) = \frac{q^n}{1 - q^n} \beta_{n-1}(aq, \lambda q) \]

and

\[ \alpha_n(\lambda, b) = \frac{q^n}{1 - q^n} \alpha_{n-1}(\lambda q, bq). \]

Iterating these we have
\[ \beta_n (a, \lambda) = \frac{n(n+1)}{2} q \frac{(q)_n}{(aq^n, \lambda q^n)} = \frac{n(n+1)}{2} g (aq^n, \lambda q^n, 0, q) \]

and

\[ \alpha_n (\lambda, b) = \frac{n(n+1)}{2} q \frac{(q)_n}{(aq^n, bq^n)} = \frac{n(n+1)}{2} g (0, \lambda q^n, bq^n, q). \]

These imply

\[ g (a, \lambda, b, q) = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} q \frac{(q)_n}{(aq^n, \lambda q^n, 0, q)} b^n \]

...(9)

and

\[ g (a, \lambda, b, q) = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} q \frac{(q)_n}{(0, \lambda q^n, bq^n, q)} a^n. \]

...(10)

Interchanging \(a\) and \(b\) and using (1), (10) becomes

\[ g (b, \lambda, a, q) = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} q \frac{(q)_n}{(aq^n, \lambda q^n, 0, q)} b^n. \]

This with (9) proves (3).

An alternate proof of (3) but dependent on the well-known Euler's formulae for the expansions of \((a + \lambda)(a + \lambda q)\ldots(a + \lambda q^{n-1})\) and of \((- bq^{n+1}) \infty\) in powers of \(\lambda\) and \(b\) respectively, can be given.

REFERENCES


