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# A BASIC HYPERGEOMETRIC TRANSFORMATION OF RAMANUJAN AND A GENERALIZATION

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We prove in an elementary and self-contained manner a basic hypergeometric identity of Ramanujan and its equivalence to a more general identity.

## 1. INTRODUCTION

The transformation

$$(-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^n}{(q)_n(-bq)_n} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{b}\right)_n b^n}{(q)_n} \qquad \ldots(1)$$

is found in the Second Notebook of Srinivasa Ramanujan[7] [vol. II, p. 194, Entry 9]. Here,

$$(a)_\infty = (a, q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a)_k = (a, q)_k = \begin{cases} \dfrac{(a)_\infty}{(aq^k)_\infty}, & k\text{: any integer,} \\ = \prod_{n=0}^{k-1} (1 - aq^n), & k\text{: positive integer,} \end{cases}$$

where $a$ and $q$ are complex numbers with $|q| < 1$. A particular case ($b = 1$, $\lambda = -a$) of (1) was first posed as an Advanced problem by L. Carlitz[4] who also

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proved the general case by employing Euler's expansion for $(a)_r$ as a polynomial in $a$. V. Ramamani[5] has given a proof of (1) by obtaining two functional relations for the right hand side. Andrews[2] has shown that (1) is a limiting case of an identity of Rogers. Ramamani and Venkatachaliengar[6] have observed that (1) can be obtained as a limiting case of an identity of Heine. More recently Adiga *et al.*[1] have obtained two other proofs of (1). One of their proofs consists in making iterated use of Heine's identity and the other in obtaining a functional relationship for the left side of (1).

In this note we consider a more general hypergeometric series

$$g(a, \lambda, b, q) = (-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} \left(-\frac{\lambda}{a}\right)_n a^n}{(q)_n (-bq)_n} \qquad \ldots(2)$$

and give a simple, self-contained approach not only to proving (1) but also to show that (1) implies a more general identity

$$(-bq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} \left(-\frac{\lambda}{a}\right)_n a^n}{(q)_n (-bq)_n} = (-aq)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} \left(-\frac{\lambda}{b}\right)_n b^n}{(q)_n (-aq)_n} \qquad \ldots(3)$$

or, what is the same

$$g(a, \lambda, b, q) = g(b, \lambda, a, q). \qquad \ldots(3')$$

The function $g(a, \lambda, b, q)$ in (2) was also considered by Bhargava and Adiga[3] while giving a simple, unified approach to proving several continued fraction expansions of Ramanujan and Hirschhorn.

## 2. Proof of the Identity (3)

We need the following Lemma and its corollary the proofs of which are simple and have been given by Bhargava and Adiga[3] earlier.

*Lemma* 1—If $|q| < 1$, then $g$ satisfies the following functional relations :

$$g(a, \lambda, b, q) - g(aq, \lambda, b, q) = aq\, g(aq, \lambda q, bq, q), \qquad \ldots(4)$$

$$g(a, \lambda, b, q) - g(a, \lambda q, b, q) = \lambda q\, g(aq, \lambda q^2, bq, q), \qquad \ldots(5)$$

$$g(a, \lambda, b, q) - g(a, \lambda, bq, q) = bq\, g(aq, \lambda q, bq, q). \qquad \ldots(6)$$

*Corollary* 1—$g(aq, \lambda, b, q) = (1 - aq + bq) + g(aq, \lambda q, bq, q) + (aq + \lambda q)\, g(aq, \lambda q^2, bq^2, q)$. ...(7)

*Theorem* 1—If $|q| < 1$, then identity (1) holds.

PROOF : Setting

$$g(0, bt, b, q) = \sum_{n=0}^{\infty} \beta_n(t, q)\, b^n \qquad \ldots(8)$$

and putting $a = 0$, $\lambda = bt$ in (7) comparing coefficients we have

$$(1 - q^n)\, \beta_n = q^n (1 + tq^{n-1})\, \beta_{n-1}, \; n = 1, 2, \ldots .$$

Iterating this and noting that $\beta_0 = g(0, 0, 0, q) = 1$, we get

$$\beta_n = \frac{q^{\frac{n(n+1)}{2}}(-t)_n}{(q)_n} = \frac{q^{\frac{n(n+1)}{2}}\left(-\frac{\lambda}{b}\right)_n}{(q)_n}.$$

Substituting this in (8) we get (1).

*Theorem* 2—If $|q| < 1$, then identity (3) holds.

PROOF : Making use of the functionalrel ations (4)-(6) we show that (1) implies (3). Substituting

$$g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \beta_n(a, \lambda)\, b^n$$

and

$$g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \alpha_n(\lambda, b)\, a^n$$

in (6) and (4) respectively and comparing coefficients we have

$$\beta_n(a, \lambda) = \frac{q^n}{1 - q^n} \beta_{n-1}(aq, \lambda q)$$

and

$$\alpha_n(\lambda, b) = \frac{q^n}{1 - q^n} \alpha_{n-1}(\lambda q, bq).$$

Iterating these we have

$$\beta_n(a, \lambda) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} \beta_0(aq^n, \lambda q^n) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(aq^n, \lambda q^n, 0, q)$$

and

$$\alpha_n(\lambda, b) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} \alpha_0(\lambda q^n, bq^n) = \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(0, \lambda q^n, bq^n, q).$$

These imply

$$g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(aq^n, \lambda q^n, 0, q) b^n \qquad \ldots(9)$$

and

$$g(a, \lambda, b, q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(0, \lambda q^n, bq^n, q) a^n. \qquad \ldots(10)$$

Interchanging $a$ and $b$ and using (1), (10) becomes

$$g(b, \lambda, a, q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q)_n} g(aq^n, \lambda q^n, 0, q) b^n.$$

This with (9) proves (3).

An alternate proof of (3) but dependent on the well-known Euler's formulae for the expansions of $(a + \lambda)(a + \lambda q)\ldots(a + \lambda q^{n-1})$ and of $(-bq^{n+1})_\infty$ in powers of $\lambda$ and $b$ respectively, can be given.

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