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TOPOLOGICAL PROPERTIES OF $LH_{a,\mu}$ -SPACES

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In this paper several different $LH_{a,\mu}$-space are defined and their topological properties are studied. Various types of operators like exponential multipliers, power multipliers dilations etc. are discussed on some of these $LH_{a,\mu}$-spaces.

1. INTRODUCTION

Tiwari[1,2] has discussed spaces of type $LH_{a,\mu}$. These spaces are used[3] to solve the equation of propagation of heat in cylindrical coordinates imposing the generalized boundary conditions.

We denote by Ω_1 and Ω_2 the open sets $(\mathbf{R}_+) \times (\mathbf{R}_+)$ and $(\mathbf{R}_-) \times (\mathbf{R}_+)$ respectively. Further the classes of infinitely differentiable functions defined on Ω_1 and Ω_2 are respectively symbolized by $E(\Omega_1)$ and $E(\Omega_2)$. We use the notation Δ for the operator $x^{-1}\dfrac{\partial}{\partial x}$. Unless otherwise specified, we choose a, μ α, β from $\mathbf{R}$ with $\alpha, \beta \geq 0$ and the letters k, g, l mean 0, 1, 2 .. . Also the notation 0° wherever it occurs in the sequel, means the number 1.

1.1. The space $LH_{a,\mu,\alpha,m}$ — For given $m > 0$ it is defined as $LH_{a,\mu,\alpha,m} = \{\varphi : \varphi \in E(\Omega_1)$,

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{\alpha t} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_{qls}(m + \delta)^k k^{k\alpha}\}$$

where δ is any number greater than 0 and C^q_{ls} is a constant depending upon δ and the function φ.

1.2. The space $LH^{\beta,n}_{a,\mu}$—For given $n > 0$, we define

$$LH_{a,\mu}^{\beta,n} = \Big\{ \varphi : \varphi \in E(\Omega_1),$$

$$\gamma_{a,k,q,l}^{\mu}(\varphi) = \sup_{\Omega_1} \Big| x^k e^{at} D_t^l \Delta^q \times \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \Big\} \Big|$$

$$\leqslant C_{kl\eta} (n + \eta)^q q^{q\beta} \Big\}$$

where the constant $C_{kl\eta}$ depends on the function φ and the number $\eta > 0$.

1.3. The space $LH_{a,\mu,\alpha,m}^{\beta,n}$ — We define this space as

$$LH_{a,\mu,\alpha,m}^{\beta,n} = \Big\{ \varphi : \varphi \in E(\Omega_1),$$

$$\gamma_{a,k,q,l}^{\mu}(\varphi) = \sup_{\Omega_1} \Big| x^k e^{at} D_t^l \Delta^q \times \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \Big\} \Big|$$

$$\leqslant C_{\delta\eta} (m + \delta)^k (n + \eta)^q k^{k\alpha} q^{q\beta} \}$$

for given $m, n > 0$ and for any $\delta, \eta > 0$. The constant $C_{\delta\eta}$ depends on the function φ.

Analogous to 1.1, 1.2 and 1.3 we define the spaces $\overset{\vee}{LH}_{a,\mu,+,m}$, $\overset{\vee}{LH}_{a,\mu}^{\beta,n}$ and $\overset{\vee}{LH}_{a,\mu,\alpha,m}^{\beta,n}$ on domain Ω_2. However, for the sake of convenience we mention the space $\overset{\vee}{LH}_{a,\mu,\alpha,m}$ in

1.4. The space $\overset{\vee}{LH}_{a,\mu,\alpha,m}$ — It is given by

$$\overset{\vee}{LH}_{a,\mu,\alpha,m} = \Big\{ \varphi : \overset{\vee}{\varphi}(t, x) = \varphi(-t, x) \in LH_{a,\mu,\alpha,m},$$

(*equation continued on p.* 345)

$$P^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_2} \Big| x^k e^{-at} \; D_t^l \; \Delta^q \times \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \Big| \leqslant C_{qls} (m + \delta)^k \; k^{k\alpha}\}$$

where m is a given positive number and δ is any number greater than 0.

Unless specified otherwise, the sapces $LH_{a,\mu,\alpha,m}$; $LH^{\beta,n}_{a,\mu}$ and $LH^{\beta,n}_{a,\mu,\alpha,m}$ will henceforth be considered equipped with their natural Hausdorff locally convex topologies to be denoted respectively by $T_{a,\mu,\alpha,m}$, $T^{\beta,n}_{a,\mu}$ and $T^{\beta,n}_{a,\mu,\alpha,m}$. These topologies are genereted by the total family of seminorms $\left\{\gamma^{\mu}_{a,k,q,l}\right\}$. The space $\overset{\vee}{LH}_{a,\mu,\alpha,m}$ is equipped with the topology $T_{a,\mu,\alpha,m}$ generated by the total family of seminorms $\left\{ P^{\mu}_{a,k,q,l} \right\}$.

2. Inductive Limit and the Space $LH_{a,\mu,\alpha}$

In this section we confine attention to the strict inductive limits of the spaces given in the previous section.

Proposition 2.1—If $m_1 < m_2$, then

$$LH_{a,\mu,\alpha,m1} \subset LH_{a,\mu,\alpha,m_2} \text{ and } T_{a,\mu,\alpha,m_1} \approx T_{a,\mu,\alpha,m2} \mid LH_{a,\mu,\alpha,m1}.$$

Proof : For $\varphi \in LH_{a,\mu,\alpha,m\,1}$

$$\gamma^{\mu}_{a,k,q,l} \; (\varphi) \leqslant C_{qls} (m_1 + \delta)^k \, k^{k\alpha} \leqslant C_{qls} (m_2 + \delta)^k \, k^{k\alpha}.$$

Thus $LH_{a,\mu,\alpha,m1} \subset LH_{a,\mu,\alpha,m^2}$. The second part is clear from the definition of topologies of these spaces.

We now define the inductive limit of the spaces $\{LH_{a,\mu,\alpha,m} : m \geqslant 1\}$ as follows:

$$LH_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m}.$$

The space $LH_{a,\mu,\alpha}$ is equipped with the strict inductive limit topology defined by the injections from $LH_{a,\mu,\alpha,m}$ to $LH_{a,\mu,\alpha}$. A sequence $\{\varphi_n\}$ in $LH_{a,\mu,\alpha}$ converges to zero if and only if $\{\varphi_n\}$ is contained is some $LH_{a,\mu,\alpha,m}$ and converges their into zero.

The following result exhibits the exponential behaviour of the members of $LH_{a,\mu,\alpha}$.

Theorem 2.2—For $\varphi \in LH_{a,\mu,\alpha}$ and $\alpha > 0$, the following inequality holds :

$$\left| D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant D_{ql} \exp \{-bx^{1/\alpha} - at\}$$

where $b = \{\alpha/e\, A^{1/\alpha}\}$, A and D_{ql} are some constants.

PROOF : Let $\varphi \in LH_{a,\mu,\alpha}$, where $\alpha > 0$.

Then $\varphi \in LH_{a,\mu,\alpha,m}$ for some m and

$$\left| x^k e^{at} D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant C_{ql\delta} (m + \delta)^k k^{k\alpha}.$$

Choose constants A and C_{ql} such that

$$C_{ql\delta} (m + \delta)^k \leqslant C_{ql} A^k k^{k\alpha}.$$

Then we have

$$\left| x^k e^{at} D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant C_{ql} A^k k^{k\alpha}.$$

Dividing both the sides by $x^k e^{at}$ and taking the infimum over k on the right-hand side we get

$$\left| D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right|$$

$$\leqslant C_{ql} e^{-at} \inf_k \frac{k^{k\alpha}}{T^k}, T = \frac{x}{A}$$

$$\leqslant C_{ql} e^{-at} \inf_k f(k), f(k) = \frac{k^{k\alpha}}{T^k} . \qquad \ldots(A)$$

Since k assumes only integral values, we find min $\{f(u) : u \geqslant 0\}$.

Using the elementary calculus method for computation of maxima and minima, we find that $\min_{u \geqslant 0} f(u)$ is attained at

$$u_0 = \frac{1}{e}(T^{1/\alpha}).$$

Thus

$$\min_{u \geqslant 0} f(u) = \exp\left[-(\alpha/e)\, T^{1/\alpha}\right]. \qquad \ldots(B)$$

For obtaining $\inf_k f(k)$, write $k_0 = [u_0] + 1$.

Let $0 \leqslant h \leqslant 1$, be such that $k_0 = u_0 + h$.

Then

$$f(k_0) = \frac{(u_0 + h)^{(u_0+h)\alpha}}{T^{u_0+h}}$$

$$= \frac{u_0^{(u_0+h)\alpha}}{(T)^{u_0+h}}\left[1 + \frac{h}{u_0}\right]^{(u_0+h)\alpha}$$

$$= f(u_0)\, \frac{\left(\frac{1}{e}\, T^{1/\alpha}\right)^{h\alpha}}{T^h}\left(1 + \frac{h}{u_0}\right)^{u_0\alpha}\left(1 + \frac{h}{u_0}\right)^{h\alpha}$$

$$\leqslant f(u_0)\left(1 + \frac{h}{u_0}\right)^{h\alpha}$$

$$\leqslant \exp\left(-\frac{\alpha}{e}\, T^{1/\alpha}\right)\left(1 + \frac{h\, e}{T^{1/\alpha}}\right)^{h\alpha}. \qquad \ldots(C)$$

We now consider two cases : (i) $T \geqslant 1$ and (ii) $T < 1$.

(i) For $T \geqslant 1$, $(T)^{-1/\alpha} \leqslant 1$ and so from (C),

$$f(k_0) \leqslant A_1 \exp\left(-\frac{\alpha}{e}\, T^{1/\alpha}\right) \qquad \ldots(D)$$

where $A_1 = (1 + he)^{h\alpha}$, a constant.

(ii) For $T < 1$, observe that

$$\min_k f(K) \leqslant 1 \leqslant \exp\left(\frac{\alpha}{e}\right) \exp\left(-\frac{\alpha}{e}\, T^{1/\alpha}\right).$$

Thus

$$\min_k f(k) \leqslant B \exp\left(-\frac{\alpha}{e}\, T^{1/\alpha}\right) \qquad \ldots(E)$$

where $B = \exp(\alpha/e)$, a constant.

Hence from (D) and (E), we get

$$\min_k f(k) < C \exp\left(-\frac{\alpha}{e} T^{1/\alpha}\right) \qquad \ldots(F)$$

for all values of T.

Also further from (A) and (F),

$$\left| D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant C_{ql}.\ c \exp\left(-\frac{\alpha}{e} T^{1/\alpha} - at\right)$$

$$\leqslant D_{ql} \exp\left[-bx^{1/\alpha} - at\right]$$

where

$$b = \{\alpha/(eA^{1/\alpha})\}.$$

This completes the proof of the theorem.

Note : Let us also point out the following inequalities obtained from (B) and (F).

$$\exp\left(-\frac{\alpha}{e} T^{1/\alpha}\right) < \inf_k f(k) < C \exp\left(-\frac{\alpha}{e} T^{1/\alpha}\right). \qquad \ldots(2.1)$$

3. Some Results on the Space $LH_{a,\mu,\alpha,m}$

In this section, we consider two equivalent topologies on $LH_{a,\mu,\alpha,m}$, which are stronger than $T_{a,\mu,\alpha,m}$ as introduced in section 1. To begin with let us introduce

$$M_p(t, x) = \exp\left[b\left(1 - \frac{1}{p}\right) x^{1/\alpha} + at \right]$$

where $b = \{\alpha/(em^{1/\alpha})\}$, $p = 2, 3, \ldots$ Then we have

Proposition 3.1— For $\varphi \in LH_{a,\mu,\alpha,m}$, one has

$$\|\varphi\|_p^{\mu} = \sup_{\substack{q \leqslant p \\ l \geqslant 0}} \left[\sup_{\Omega_1} M_p(t, x) \left| D_t^l \ \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \right] < \infty$$

and $\|.\|_p^{\mu}$ defines a norm on $LH_{a,\mu,\alpha,m}$.

Proof : By Theorem 2.2, for $\varphi \in LH_{a,\mu,\alpha,m}$, we have

$$\|\varphi\|_p^{\mu} \leqslant \sup_{\substack{q\leqslant p \\ l\geqslant 0}} \sup_{\Omega_1} \left[D_{ql} \exp\left\{ b\left(1-\frac{1}{p}\right)x^{1/\alpha} + at \right\} \right.$$

$$\left. \times \exp\{-bx^{1/\alpha} - at\}\right]$$

$$= \sup_{\substack{q\leqslant p \\ l\geqslant 0 \\ <\infty}} \sup_{\Omega_1} D_{ql} \exp\left(-\frac{b}{p} x^{1/\alpha} \right).$$

Hence $\|.\|_p^{\mu}$ is well defined for each ϕ on $LH_{a,\mu,\alpha,m}$. Clearly it is a norm on this space. This completes the proof of the proposition.

Definition 3.2—Let D_{μ} denotes the family of norms $\{\|\phi\|_p^{\mu}$, $p = 2, 3 \ldots\}$ as defined in Proposition 3. 1 and T_{μ} be the topology generated by D_{μ}.

On $LH_{a,\mu,\alpha,m}$ we can define another set of norms associated with the original definition (1.1) of this space, namely

$$\|\varphi\|_{q,\delta}^{\mu} \sup_{k,l} \sup_{\Omega_1} \frac{\left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right|}{(m+\delta)^k k^{k\alpha}} .$$

Let $T_{\mu,\delta}$ be the topology generated by the family of norms $D_{\mu,\delta} = \left\{ \|\varphi\|_{q,\delta}^{\mu} \right.$ $q = 0, 1, \ldots$ are $\delta = 1, 2/2, \ldots \left. \right\}$. Then we have

Theorem 3.3—The topologies T_{μ} and $T_{\mu,\delta}$ are equivalent and are finer than $T_{a,\mu,\alpha,m}$.

PROOF: For showing $T_{\mu} \approx T_{\mu,\delta}$, we need prove that D_{μ} and $D_{\mu,\delta}$ are equivalent.

Consider a member $\|\cdot\|_{q,\delta}^{\mu}$ in $D_{\mu,\delta}$,

where

$$\|\phi\|_{q,\delta}^{\mu} = \sup_{k,l} \sup_{\Omega_1} \frac{\left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right|}{(m+\delta)^k k^{k\alpha}} .$$

Let us note

$$\sup_k \left[\frac{x^k}{(m+\delta)^k k^{k\alpha}} \right] = \frac{1}{\inf f(k)}$$

where

$$f(k) = \frac{k^{k\alpha}}{T^k}, \; T = \left(\frac{x}{(m+\delta)} \right).$$

But

$$\frac{1}{\inf\limits_{k} f(k)} \leqslant \exp\left(\frac{\alpha}{e} T^{1/\alpha} \right)$$

by (2.1). Hence

$$\|\phi\|^{\mu}_{q,\delta} \leqslant \sup_{l} \sup_{\Omega_1} \left[\exp\left\{ \frac{\alpha}{e} \left(\frac{x}{(m+\delta)} \right)^{1/\alpha} + at \right\} \right] \times D_t^l \, \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{1} \right)} \phi \right\} \Big|$$

$$\leqslant \sup_{l} \sup_{\Omega_1} \exp\left\{ \frac{\alpha}{em^{1/\alpha}} \frac{x^{1/\alpha}}{\left(1 + \frac{\delta}{m} \right)} + at \right\} \times D_t^l \, \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2} \right)} \varphi \right\} \Big|.$$

Choose $p_0 \in \mathbb{N}$ such that $\dfrac{1}{\left(1 + \frac{\delta}{m} \right)^{1/\alpha}} < 1 - \dfrac{1}{p_0}$.

Then for

$$b = \left(\frac{\alpha}{em^{1/\alpha}} \right).$$

$$\|\phi\|^{\mu}_{q,\delta} \leqslant \sup_{l} \sup_{\Omega_1} \exp\left[b \left(1 - \frac{1}{p_0} \right) x^{1/\alpha} + at \right] \times \left| D_t^l \Delta^q \left\{ x^{\left(-\frac{2\mu+1}{2} \right)} \phi \right\} \right|$$

$$\leqslant \sup_{l} \sup_{\Omega_1} M_{p_0}(t, x) \left| D_t^l \; \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2} \right)} \phi \right\} \right|$$

(*equation continued on p.* 351)

$$\leqslant \sup_{\substack{r\leqslant p\\ l\geqslant 0}} \sup_{\Omega_1} M_p(t,x) \left| D_t^l \Delta^r \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right|$$

$$= \|\phi\|_p^{b_p}$$

where $p = \max(p_0, q)$.

Conversely for fixed $p \geqslant 2$ we have

$$\|\varphi\|_p^{b_p} = \sup_{\substack{q\leqslant p\\ l\geqslant 0}} \sup_{\Omega_1} \left(\exp\left\{ b\left[\left(1 - \frac{1}{p}\right) x^{1/\alpha} + at \right\} \right. \right.$$

$$\left. \left. \times \left| D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right| \right] . \qquad \ldots(G)$$

Again using (2.1) we get

$$\exp\left[\frac{\alpha}{e} \frac{(x)^{1/\alpha}}{(m+\delta)^{1/\alpha}} \right] \leqslant C_2 \cdot \sup_k \left[\frac{x^k}{(m+\delta)^k k^{k/\alpha}} \right].$$

Since $0 < m \left[\frac{1}{\left(1 - \frac{1}{p}\right)^{\alpha}} - 1 \right]$, we can choose $\delta \rhd 0$ so that

$$1 - \frac{1}{p} < \frac{1}{\left(1 + \frac{\delta}{m}\right)^{1/\alpha}} .$$

Hence

$$\exp\left[b \left(1 - \frac{1}{p}\right) x^{1/\alpha} \right] < \exp\left[b \frac{m^{1/\alpha} x^{1/\alpha}}{(m+\delta)^{1/\alpha}} \right] \qquad \ldots(H)$$

Consequently from (G) and (H).

$$\|\varphi\|_p^{b_p} \sup_{\substack{q\leqslant p\\ l\geqslant 0}} \sup_{\Omega_1} \left[\exp\left\{ \frac{bm^{1/\alpha} x^{1\alpha}}{(m+\delta)^{1/\alpha}} \right\} \right.$$

$$\left. \times \left| e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right| \right], \text{ by} \qquad \ldots(I)$$

$$\leqslant C_2 \sup_{q<p} \|\varphi\|_{q;\delta}^{b_p} .$$

This completes our proof

4. On the Nontriviality of the LH-spaces

Lemma 4.1—For $\varphi \in LH_{a,\mu,\alpha,m}$,

$$\Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} = \sum_{j=0}^{q} a_j \, x^{-\left(\frac{2\mu+1}{2}\right)-q-j} \times D_x^{q-1} \varphi \qquad \ldots(4.1)$$

where a_j's are some constants.

Proof : Proof is immediate by induction.

Theorem 4.2—The space $D(\Omega_1)$ is a subspace of $LH_{a,\mu,\alpha,m}$, such that the injection map form $D(\Omega_1)$ to $LH_{a,\mu,\alpha,m}$ is continuous i.e. $T_{a,\mu,\alpha,m} \mid D(\Omega_1) \subset T_{\Omega_1}$.

Proof : For $\varphi(t, x) \in D(\Omega_1)$, set

$$L = \sup \{x : (t, x) \in \sup p(\phi)\} :$$

$$C_{ql} = \sup_{\Omega_1} \left| e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right| .$$

Then

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right| \leqslant C_{ql} L^k$$

$$\leqslant C_{ql} \left(\frac{L}{(m+\delta) k^\alpha} \right)^k (m+\delta)^k k^{k\alpha}$$

$$< D_{ql\delta} (m+\delta)^k k^{k\alpha}.$$

Above inequality clearly implies that $\varphi \in LH_{a,\mu,\alpha,m}$, consequently $D(\Omega_1) \subset LH_{a,\mu,\alpha,m}$.

To prove the continuity of the injection map, consider a sequence $\{\phi_n\}$ in $D(\Omega_1)$, converging to zero. Using Lemma 4.1,

$$\gamma^{\mu}_{a,k,q,l}(\phi_n) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \sum_{j=0}^{q} a_j \, x^{-\left(\frac{2\mu+1}{2}\right)-q-j} \times D_x^{q-j} \phi_n \right|$$

(equation continued on p. 353)

$$\leqslant \sum_{j=0}^{q} \sup_{\Omega_1} \left| e^{at}\, a_j\, x^{k-\left(\frac{2\mu+1}{2}\right)-q-j}, \right.$$

$$\left. \times D_t^l \; D_x^{q-j} \; \phi_n \right|$$

$$\leqslant \sum_{j=0}^{q} \sup_{\Omega_1} \left| \left\{ e^{at}\, a_j\, x^{k-\left(\frac{2\mu+1}{2}\right)-q-j} \right\} \right.$$

$$\left. \times \{D^{l+q+j}\, \phi_n\} \right| \qquad \ldots(4.2)$$

Now $\phi_n \in D(\Omega_1)$ so that we can find a compact subset K_m (say) of Ω_1 such that $\phi_n \in D(K_m)$ for each $n \geqslant 1$ and $\phi_n \to 0$ in $D(K_m)$. If

$$M_j = \sup_{K_m} \left| e^{at}\, a_j\, x^{k-\left(\frac{2\mu+1}{2}\right)-q-j} \right|.$$

Then from (4.2)

$$\gamma^{m}_{a,k,q,l}(\phi_n) \leqslant \sum_{j=0}^{q} M_j \sup_{\substack{k \\ m}} \left| D^{l+q-j}\, \phi_n \right|$$

$$\to 0 \text{ as } n \to \infty.$$

Hence $\phi_n \to 0$ in $LH_{a,\mu,\alpha,m}$ and therefore $T_{a,\mu,\alpha,m} \mid D(\Omega_1) \subset T_{a1}$. This completes our proof.

The following example illustrates the proper containment of $D(\Omega_1)$ in $LH_{a,\mu,\alpha,m}$.

Example 4.3—Consider a function $\phi : \Omega_1 \to \mathbb{C}$ defined by

$$\phi(t, x) = x^{\frac{(2\mu+1)}{2}} \exp\left\{ -at - \frac{Bx^2}{2} \right\}, (m+\delta)^2 B \geqslant 1.$$

We show that $\phi \in LH_{a,\mu,\alpha,m}$. Indeed.

$$\sup_{\Omega_1} \left| x^k\, e^{at}\, D_t^l \;\; \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \right\} \right|$$

$$= \sup_{\Omega_1} \left| x^k\, a^l\, B^q \exp\left(-\tfrac{1}{2} Bx^2\right) \right.$$

$$\leqslant \sup_{\Omega_1} \left| x^k a^l B^q \exp\left(-\frac{x^2}{2(m+\delta)^2}\right)\right|$$

$$\leqslant \sup_{\Omega_1} C_1 B^q (m+\delta)^k \left| u^k \exp\left(-\frac{u^2}{2k}\right)\right|, u = \frac{x}{m+\delta}$$

$$\leqslant C_1 B^q (m+\delta)^k \sup_{\Omega_1} \left| \left\{u^2 \exp\left(-\frac{u^2}{2k}\right)\right\}^{k/2}\right|$$

$$\leqslant C_1 B^q (m+\delta)^k k^{k/2}.$$

Thus $\phi \in LH_{a,\mu,\alpha,m}$.

It is clear that ϕ has no compact support and hence $\phi \notin D(\Omega_1)$. This proves $D(\Omega_1) \subsetneq LH_{a,\mu,\alpha,m}$.

It is easy to see that the function ϕ of Example 4.3 also belongs to the space $LH^{\beta,n}_{a,\mu,\alpha,m}$ This proves the nontriviality of the space $LH^{\beta,n}_{a,\mu,\alpha,m}$.

For nontriviality of the space $LH^{\beta,n}_{a,\mu}$ we have the following example :

Example 4.4—The function $\phi : \Omega_1 \to \mathbb{C}$ defined by

$$\phi(t, x) = x^{\left(\frac{2\mu+1}{2}\right)} \exp\left\{-at - \frac{1}{2} b x^2\right\}, b \geqslant 1$$

belongs to the space $LH^{\beta,n}_{a,\mu}$.

Proof is easy and so omitted.

5. Some Operations in $LH_{a,\mu}$-spaces

(a) *Exponential Multiplier*

Proposition 5.1—Let $\sigma \in \mathbf{R}$ and $\varphi \in LH_{a-\sigma,\mu,\alpha,m}$.

If $\psi(t, x) = \exp(-\sigma t)\, \phi(t, x)$, then $\psi \in LH_{a,\mu,\alpha,m}$.

Proof : For $\gamma^{\mu}_{a,k,q,l} \in D_{a,\mu,\alpha,m}$ denotes the family of all semi-norms generating the topology on $(LH_{a,\mu,\alpha,m})$, consider

$$\gamma^{\mu}_{a',k,q,l}(\psi) = \sup_{\Omega_1} \Big| x^k e^{at} D_t^l \Delta^q \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \times (e^{-\sigma t}\phi) \Big\} \Big|$$

$$= \sup_{\Omega_1} \Big| x^k e^{at} \Delta^q \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \sum_{\nu=0}^{l} \binom{l}{\nu} \times (-\sigma)^{l-\nu} e^{-\sigma t} D_t^{\nu} \phi \Big\} \Big|$$

$$\leqslant \sum_{\nu=0}^{l} a_\nu \sup_{\Omega_1} \Big| x^k e^{(a-\sigma)t} D_t^{\nu} \Delta^q \times \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \phi \Big\} \Big|, \; a_\nu = \binom{l}{\nu} (-\sigma)^{l-\nu}$$

$$\leqslant \sum_{\nu=0}^{l} a_\nu \; C_{q\nu\delta} (m+\delta)^k k^{k\alpha}. \qquad \ldots(5.1)$$

Since $\varphi \in LH_{a-\sigma,\mu,\alpha,m}$. Thus $\psi \in LH_{a,\mu,\alpha,m}$.

Proposition 5.2—The exponential multiplier

$E : (LH_{a-\sigma,\mu,\alpha,m};\ T_{a-\sigma,,\alpha,m}) \to (LH_{a,\mu,\alpha,m};\ T_{a,\mu,\alpha,m})$is a topological isomorphism.

PROOF : E is injective, since exponential function never assumes the value 0 in Ω_1.

Also $\psi \in LH_{a-\sigma,\mu,\alpha,m};\ \psi(t, x) = e^{\sigma t}\phi$

whenever $\phi \in LH_{a,\mu,\alpha,m}$. Hence E is surjective too.

For the continuity of E, note the following from (5.1)

$$\gamma^{\mu}_{a',k,q,l}(e^{-\sigma t}\phi) \leqslant \sum_{\nu=0}^{l} a_\nu \; \gamma^{\mu}_{a-\sigma,k,q,\nu}(\phi).$$

The inverse mapping $E^{-1}(\phi) = \psi,\ \psi(t, x) = e^{\sigma t}\phi(t, x)$ can be proved to be $1 - 1$ and onto in a similar way. This ends our proof.

(b) *Power Multiplier*

Proposition 5.3—For $\phi \in LH_{a,\mu,\alpha,m}$ and $n \in \mathbb{Z}$, if $\psi(t, x) = x^n \phi(t, x)$, $(t, x) \in \Omega_1$, then $\psi \in LH_{a,\mu+n,\alpha,m}$.

PROOF : Observe that

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+2n+1}{2}\right)} x^n \phi \right\} \right|$$

$$= \gamma^{\mu}_{a,k,q,l} (\phi)$$

$$\leqslant C_{qls} (m + \delta)^k k^{k\alpha}.$$

Since $\phi \in LH_{a,\mu,\alpha,m}$. Thus $\phi \in LH_{a,\mu+n,\alpha,m}$. Moreover, it is clear from the above that

$$\gamma^{\mu+n}_{a,k,q,l} (\psi) = \gamma^{\mu}_{a,k,q,l} (\phi). \qquad \ldots(5.2)$$

The above results help us to have

Proposition 5.4—The power multiplier operator

$P : LH_{a,\mu,\alpha,m} \to LH_{a,\mu+n,\alpha,m}$ define by $P\varphi = \psi$, where $\psi(t, x) = x^n \varphi(t, x)$, $\varphi \in LH_{a,\mu,\alpha,m}$ is a topological isomorphism.

PROOF : Clearly, P is well defined linear and injective. It is surjective because for ψ in $LH_{a,\mu+n,\alpha,m}$, $\varphi(t, x) = x^{-n} \psi(t, x)$ is a member of $LH_{a,\mu,\alpha,m}$ such that $P(\varphi) = \psi$.

Continuity of P is immediate from (5.2). The inverse mapping $P^{-1} : LH_{a,\mu+n,\alpha,m} \to LH_{a,\mu,\alpha,m}$ is given by $p^{-1}(\psi) = \varphi$, $\varphi(t, x) = x^{-n} \psi(t, x)$ and so it is clearly linear bijective and continuous. This completes our proof.

(c) *Smooth Multiplier Operator*

Proposition 5.5—Let $a, b \in \mathbf{R}$ with $a < b$ and $\theta \in \theta_M$, where θ_M is the space as defined in (Zemanian[4] p. 61). Then the product $\theta \varphi \in LH_{a,\mu,\alpha,m}$, for each φ in $LH_{b,\mu,\alpha,m}$.

PROOF : Since $a < b$ and $\theta \in \theta_M$, for some constant c_{lv} we get

$$\left| \left\{ e^{(a-b)t} (1 + t^2)^{N_{l-v}} \right\} \left\{ \frac{D^{l-v} \theta}{(1 + t^2)^{N_{l-v}}} \right\} \right| < C_{lv}. \qquad \ldots(5.3)$$

Consider

$$\sup_{\Omega_1} \left| x^k \; e^{at} \; D_t^l \; \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \; \theta(t) \; \varphi(t, x) \right\} \right|$$

$$= \sup_{\Omega_1} \left| x^k \, e^{bt} \, \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \, e^{(a-b)t} \right.\right.$$

$$\left.\left. \times \sum_{v=0}^{l} \binom{l}{v} \left(D_t^{l-v} \, \theta \right) \left(D_t^{v} \, \varphi \right) \right\} \right|$$

$$= \sup_{\Omega_1} \left| \sum_{l=0}^{v} \binom{l}{v} \left[x^k \, e^{bt} \, D_t^v \; \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \; \varphi \right\} \right] \right.$$

$$\left. \times \left[e^{(a-b)t} \, (1 + t^2)^{N_{l-v}} \frac{D_t^{l-v} \, \theta(t)}{(1 + t^2)^{N_{l-v}}} \right] \right|$$

$$\leqslant \sum_{l=0}^{v} \binom{l}{v} C_{lv} \, C_{qv} \, (m + \delta)^k \, k^{k\alpha}$$

by (5.3) and using the definition of members of $LH_{b,\mu,\alpha,m}$.

Thus $\theta \varphi \in LH_{a,\mu,\alpha,m}$ and also

$$\gamma^{\mu}_{a,k,q,l} \; (\theta \varphi) \leqslant \sum_{l=0}^{v} \binom{l}{v} C_{lv} \; \gamma^{\mu}_{b,k,q,l} \; (\varphi). \qquad \ldots(5.4)$$

This proposition lead us to :

Proposition 5.6—The smooth multiplier operator

$$S : (LH_{b,\mu,\alpha,m}, \; T_{b,\mu,\alpha,m}) \to (LH_{a,\mu,\alpha,m}; \; T_{a,\mu,\alpha,m})$$

defined by $S \varphi = \theta \varphi$, $\theta \in \theta_M$, $\varphi \in LH_{a,\mu,\alpha,m}$ is a continuous linear transformation.

PROOF : S is clearly well defined and linear. Moreover, it is continuous inequality (5.4).

(d) *Dilation Operation*

It is proved (Tiwari[2], p. 374) that the operation $\varphi(t, x) \to \varphi(wt, \lambda x)$ transforms the space $LH^{\beta,B}_{a,\mu}$ (respectively $LH^{\beta,B}_{a',\mu',\alpha',A}$) into the space $LH^{\beta,\lambda B}_{aw',\mu'}$ (respectively $LH^{\beta,B}_{aw,\mu',\alpha',A/\lambda}$).

The above results are not correct. The correct results are given below.

Proposition 5.7—For any pair (w, λ) of strictly positive real numbers and $\varphi \in LH^{\beta,n}_{a',\mu'}$, the function $\psi \in LH^{\beta,\lambda^2 n}_{aw,\mu'}$ $\psi(t, x) = \varphi(wt, \lambda x)$.

PROOF : For ψ as above, consider

$$\sup_{\Omega_1} \left| x^k e^{awt} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \psi(t, x) \right\} \right|$$

$$= \sup_{\Omega_1} \left| v^k e^{au} D_u^l (v^{-1} D_v)^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(u, v) \right\} \right|$$

$$\times \frac{w^l \lambda^{2q}}{\lambda^k} + \frac{2\mu+1}{2}, \text{ for } u = wt, v = \lambda x$$

$$\leqslant C'_{kl\eta} (\lambda^2 (n + \eta))^q q^{q\beta} \qquad \ldots(5.5)$$

$$\leqslant C'_{kl\eta} (\lambda^2 n + \eta')^q q^{q\beta}, \eta' > 0; \text{ by definition of } LH^{\beta,n}_{a,\mu} \text{ and } \varphi \in LH^{\beta,n}_{a',\mu'}.$$

Consequently $\varphi(wt, \lambda x) \in LH^{\beta,\lambda^2 n}_{aw',\mu'}$ and our proof is complete.

Proposition 5.8—The dilation operator

$$R : \left(LH^{\beta,n}_{a',\mu'} ; T^{\beta,n}_{a',\mu'} \right) \to \left(LH^{\beta,\lambda^2 n}_{aw',\mu'} ; T^{\beta,\lambda^2 n}_{aw',\mu'} \right)$$

with $R\varphi = \psi$ is a topological isomorphism.

PROOF : To prove $1 - 1$ ness of R, let $R\varphi = 0$. Then $\varphi(u, v) = 0$ for $u = wt$, $v = \lambda x$, $(t, x) \in \Omega_1$. As $w, \lambda > 0$, clearly $\varphi(t, x) = 0$, for all $(t, x) \in \Omega_1$.

Hence R is one-one.

R is clearly surjective, for $\psi \in LH^{\beta,n}_{\alpha,\mu}$

weenever $\varphi \in LH^{\beta,\lambda^2 n}_{\alpha w,\mu}$, where $\psi(t, x) = \varphi\left(\frac{t}{w}, \frac{x}{\lambda}\right)$.

The continuity of R follows from eqn. (5.5).

The inverse mapping $R^{-1}: LH^{\beta,\lambda^2 n}_{\alpha,w,\mu} \to LH^{\beta,n}_{\alpha,\mu}$, $R^{-1}(\psi) = \varphi$, $\varphi(t, x) = \psi(w^{-1}t, \lambda^{-1}x)$; of R is a mapping of type R itself and so it is $1-1$, onto and continuous. This completes the proof of the theorem.

Analogously we prove;

Proposition 5.9—For any pair (w, λ) of strictly positive real numbers, the dilation operator R :

$$\left(LH^{\beta,n}_{\alpha',\mu,\alpha,m}, T^{\beta,n}_{\alpha',\mu,\alpha,m}\right) \to \left(LH^{\beta,\lambda^2 n}_{\alpha w,\mu,\alpha,\frac{m}{\lambda}}; T^{\beta',\lambda^2 n}_{\alpha w,\mu,\alpha,\frac{m}{\lambda}}\right)$$

with $R\varphi = \psi$, is a topological isomorphism.

References

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4. A. H. Zemanian, *Generalized Integral Transformations*. Interscience. New York, 1968.