ON SOLAR PROPERTIES OF SETS AND RADIAL CONTINUITY 
OF SET-VALUED \( f \)-PROJECTIONS 

P. GOVINDARAJULU

Department of Mathematics, Sri Venkateswara University, Tirupati

AND

D. V. PAI

Department of Mathematics, Indian Institute of Technology
Powai, Bombay 400076

(Received 8 October 1984; after revision 9 September 1985)

Let \( V \) be a nonempty closed subset of a separated locally convex space \( X \). Given a continuous convex function \( f \) defined on \( X \), one defines here the so-called \( f \)-projection \( P_f \), supported on the set \( V \). In this exposition, radial semi-continuity concepts are studied for the set-valued mapping \( P_f \) and several relationships between properties of the set \( V \) and continuity of the mapping \( P_f \) are explored.

1. Introduction

Let \( X, Y \) be a pair of linear spaces put in duality by a separating bilinear form \( \langle \cdot, \cdot \rangle \) and equipped with locally convex topologies compatible with the pairing. Let \( f \) be a continuous convex function defined on \( X \) and satisfying \( f(0) = 0 \). Given a nonempty closed subset \( V \) of \( X \) and \( x \in X \), let

\[
fv(x) = \inf \{ f(x - v) : v \in V \}
\]

and

\[
P_f(x) = \{ v \in V : f(x - v) = fv(x) \} \text{ (possibly void)}.
\]

The set-valued mapping \( P_f \) is called \( f \)-projection supported on \( V \). \( V \) is said to be \( f \)-proximinal (resp. \( f \)-Chebyshev) if \( P_f(x) \) is nonempty (resp. a singleton) for each \( x \) in \( X \). In case \( X \) is a normed space with the norm topology and \( f \) is the given norm, there has been a lot of interest (cf. Vlasov\textsuperscript{11} for an excellent survey) in studying properties of the supporting set related to its \( f \)-projection (the so-called metric projection in this case). In case \( f \) is sublinear, properties of sets related to \( f \)-projections have been investigated earlier\textsuperscript{7-9}. These have been further extended\textsuperscript{10} to the case when \( f \) is a
continuous convex function satisfying \( f(\theta) = 0 \). Brosowski and Deutsch\(^{1-3}\) have introduced radial continuity concepts for metric projections which are sharper than the usual upper and lower semicontinuity in relation to properties of the supporting set. The principal aim of the present exposition is to extend to \( f \)-projections of Pai and Govindarajulu\(^{10}\), these radial continuity concepts and to investigate properties of the supporting set related to these. Brosowski\(^{4-5}\) has also extended and applied in a different way radial continuity concepts to the problem of parametric semi-infinite optimization.

2. Preliminary Results and \( f \)-suns

Let \( f : X \to \mathbb{R} \) be a continuous convex function satisfying \( f(\theta) = 0 \). For \( r \) in \( \mathbb{R} \), \( r > 0 \), let \( S_r := \{ x \in X : f(x) \leq r \} \) denote the sub-level subset of \( f \). \( S_r \) is a convex absorbing set containing the origin \( \theta \) in its interior. Let \( P_r(x) := \inf \{ \lambda > 0 : x \in \lambda S_r \} \) \( (x \in X) \) denote the Minkowski gauge of \( S_r \). Then \( P_r \) is a non-negative continuous sublinear function.

Recall (cf. Castaing and Valadier\(^{1}\)) that \( f \) is said to be inf-compact if the sub-level sets \( S_r \) are compact for each \( r \) in \( \mathbb{R} \). The set \( V \) is said to be an \( f \)-sun (resp. a strict \( f \)-sun) if for each \( x \in X \), \( v \) in \( P_f(x_o) \) holds for some (resp. each) element \( v \) in \( P_f(x) \) and each \( a \geq 1 \), where \( x_o = v + a(x - v) \).

**Proposition 2.1**—Let \( f \) be a non-negative continuous convex function satisfying \( f(x) = 0 \) if and only if \( x = \theta \) and the property

\[
(*) \quad \text{There exists a continuous bijection } \psi : \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{(} \mathbb{R}_+ \text{ = non-negative reals) such that}
\]

\[
f(ax) = \psi(a)f(x) \quad (a \geq 0 \text{ and } x \text{ in } X).
\]

Let \( V \) be \( f \)-proximinal. If \( V \) is an \( f \)-sun (resp. a strict \( f \)-sun), then \( V \) is a \( P_r \)-sun (resp. a strict \( P_r \)-sun) for each \( r > 0 \). Conversely, if \( V \) is a \( P_r \)-sun (resp. a strict \( P_r \)-sun) for some \( r > 0 \), then \( V \) is an \( f \)-sun (resp. a strict \( f \)-sun).

**Proof**: By Proposition 2.5 (i) of Govindarajulu and Pai\(^{10}\), \( P_f = P_{P_r} \) for each \( r > 0 \). If \( v \) is an \( f \)-sun (resp. a strict \( f \)-sun), then for each \( x \in X \), \( v \) in \( P_f(x_o) = P_{P_r}(x_o) \) for some (resp. each) element \( v \) in \( P_f(x) = P_{P_r}(x) \) and each \( a \geq 1 \) and \( r > 0 \). Hence, \( V \) is a \( P_r \)-sun (resp. a strict \( P_r \)-sun) for each \( r > 0 \). The proof of the remaining part is identical.

**Theorem 2.2**—Let \( P \) be a continuous, inf-compact sublinear function and let \( V \) be \( p \)-Chebyshev. Then \( V \) is a \( p \)-sun.

**Proof**: See Pai and Govindarajulu\(^{10}\).
Theorem 2.3—Let \( f \) be a non-negative, continuous convex function which satisfies

(i) \( f(x) = 0 \) if and only if \( x = \theta \);

(ii) the property (*) of Proposition 2.1

(iii) \( f \) is inf-compact.

Then each \( f \)-Chebyshev set is an \( f \)-sun.

PROOF: Let \( V \) be a closed \( f \)-Chebyshev set. In view of (i), it suffices to consider \( x \) in \( X \) such that \( f_\nu(x) = r > 0 \). By Proposition 2.5 (i) of Govindarajulu and Pai\(^\circ\), \( P_f = P_{r'} \) and \( V \) is \( P_f \)-Chebyshev. Also, by (iii) \( P_r \) is inf-compact and hence, \( V \) is \( P_r \)-sun by the last theorem. By Proposition 2.1, \( V \) is an \( f \)-sun.

Recall that \( V \) is said to be \( b \)-compact Govindarajulu and Pai\(^7\) if for each \( x \) in \( X \), there exists \( b \) in \( \mathbb{R} \) such that \( b > f_\nu(x) \) and \( (x - V) \cap S_b \) is compact.

Theorem 2.4—Let \( f \) be a non-negative, continuous convex function which satisfies

(i) \( f(x) = 0 \) if and only if \( x = \theta \);

(ii) \( f(-x) = f(x), \ x \in X \);

(iii) the property (*) of Proposition 2.1.

If \( V \) is a \( b \)-compact, \( f \)-Chebyshev set, then \( V \) is an \( f \)-sun.

PROOF: In view of (i), it suffices to consider \( x \) in \( X \) such that \( f_\nu(x) = r > 0 \). Since \( V \) is \( b \)-compact, there is \( a \) \( b \) > \( r > 0 \) such that \( (x - V) \cap S_b \) is compact. By property (*), choose \( 0 < a < 1 \) such that \( \psi(a) = r/b \).

By Lemma 2.2 of Govindarajulu and Pai\(^\circ\), \( P_b = aP_r \) and \( S_b = \{x : P_b(x) \leq 1/a\} = \{x : P_r \leq 1/a\} \). Since \( 1 = \inf\{P_f(x - v) : v \in V\} < 1/a, V \) is \( b \)-compact for \( P_r \).

Also, by (ii), \( P_r(-x) = P_r(x), x \in X \). By Theorem 3.1 of Govindarajulu and Pai\(^7\), \( V \) is a \( P_r \)-sun and by Proposition 2.1, \( V \) is an \( f \)-sun.

Theorem 2.5—Let \( f \) be as in Theorem 2.3. If \( V \) is an \( f \)-proximinal set supporting a convex-valued \( f \)-projection \( P_f \), then \( V \) is an \( f \)-sun.

PROOF: It suffices to consider \( x \) in \( X \) such that \( f_\nu(x) = r > 0 \). By Proposition 2.5 (i) of Govindarajulu and Pai\(^\circ\), \( P_f = P_{P_{r'}} \). Now \( P_{r'} \) is inf-compact, satisfies \( P_{r'}(x) = 0 \) if and only if \( x = \theta \) and furthermore, \( P_{P_{r'}} \) is convex-valued. Therefore, by Theorem 2.6 of Govindarajulu and Pai\(^8\), \( V \) is a \( P_r \)-sun and by Proposition 2.1, \( V \) is an \( f \)-sun.
3. **Radial Continuity of $f$-Projection**

**Definition 3.1**—Let $V$ be a closed subset of $X$ and let $x_0$ in $X$. $P_f$ is said to be outer radially lower semi-continuous (resp. inner radially lower semi-continuous), abbr. ORL-continuous (resp. IRL-continuous) at $x_0$ if, for each $v_0$ in $P_f(x_0)$ and each open set $W$ such that $P_f(x_0) \cap W \neq \emptyset$, there exists a neighbourhood $U$ of $x_0$ in the semi-ray $\sigma(x_0) = \{v_0 + a(x_0 - v_0) : a \geq 1\}$ (resp. in the line segment $[v_0, x_0]$) such that $P_f(x) \cap W \neq \emptyset$ for every $x$ in $U$. $P_f$ is said to be ORL-continuous (resp. IRL-continuous) if it is ORL-continuous (resp. IRL-continuous) at each point of $X$.

We note that $P_f$ is ORL-continuous (resp. IRL-continuous) at $x_0$ if and only if the restriction of $P_f$ to $\sigma(x_0)$ (resp. to $[v_0, x_0]$) is l.s.c. for each $v_0$ in $P_f(x_0)$. Employing this remark and the known criteria for lower semi-continuity of multi-functions enables one to obtain easily:

**Proposition 3.2**—The following statements are equivalent

1. $P_f$ is ORL-continuous (resp. IRL-continuous) at $x_0$;
2. For each $v_0, v_1$ in $P_f(x_0)$ and each neighbourhood $W$ of origin in $X$, there exists a neighbourhood $U$ of $x_0$ in $\sigma(x_0)$ (resp. in $[v_0, x_0]$) and an element $v$ in $P_f(x)$ for each $x$ in $U$ such that $v_1$ in $v + W$;
3. Given $v_0, v_1$ in $P_f(x_0)$ and a net $\{x_a\} \subseteq \sigma(x_0)$ (resp. $[v_0, x_0]$) such that $x_a \to x_0$, there exists a subnet $\{x_{a_b}\}$ of $\{x_a\}$ and a net $\{v_b\}$ with $v_b$ in $P_f(x_0)$ such that $v_b \to v_1$.

**Proposition 3.3**—If $f$ is sublinear and $V$ is an $f$-sun, then $P_f$ is ORL-continuous.

**Proof:** Let $x_0 \in X$. $V$ being an $f$-sun, there exists $v_0$ in $P_f(x_0)$ such that $v_0$ in $P_f(x_a)$ for every $x_a$ in $\sigma(x_0)$. We assert that $P_f(x_0) = \cap \{P_f(x_a) : a \geq 1\}$, from which the lemma would follow. Indeed let $v_1 \in P_f(x_0)$ and $x_a = v_0 + a(x_0 - v_0)$ for $a > 1$. Then

$$f(x_a - v_1) = f((x_0 - v_1) + (a - 1)(x_0 - v_0))$$

$$\leq f(x_0 - v_1) + (a - 1)f(x_0 - v_0)$$

$$= af(x_0 - v_0) = f(x_a - v_0),$$

whence

$v_1$ in $P_f(x_a)$.

**Theorem 3.4**—Let $f$ be as in Proposition 2.1 and let $V$ be $f$-proximinal. If $V$ is an $f$-sun, then $P_f$ is ORL-continuous.

**Proof:** It suffices to consider $x$ in $X$ such that $f_V(x) = r > 0$. Then $P_f = P_{P_f}$ and $V$ is a $p_r$-sun, whence by last lemma $P_{P_f}$ and hence, $P_f$ is ORL-continuous.
**Theorem 3.5**—Assume that $f$ satisfies property (*) of Proposition 2.1. If $P_f(x_0)$ is convex, then $P_f$ is IRL-continuous at $x_0$.

**Proof**: If $P_f(x_0) = \phi$, the result is trivial. Let $v_0, v_1$ be in $P_f(x_0)$ and consider $x_a = v_0 + a(x_0 - v_0)$, for $0 < a < 1$. Then $x_a \in [v_0, x_0]$, $v_a = v_0 + a(v_1 - v_0)$ in $P_f(x_0)$ and for $a \to 1$, $v_a \to v_1$. Also,

$$f(x_a - v_0) = f(a(x_0 - v_1)) = \psi(a)f(x_0 - v_1)$$

$$= \psi(a)f(x_0 - v_0) = f(x_a - v_0).$$

Therefore, $v_n \in P_f(x_a)$ and by Proposition 4.2 (3), $P_f$ is IRL-continuous at $x_0$.

**Corollary 3.6**—If $f$ satisfies property (*) of Proposition 2.1 and $P_f$ is convex-valued, then $P_f$ is IRL-continuous.

**Remark**: The converse of Theorem 3.5 is false. For example, let $X = \mathbb{R}^2$, $f(x) = \max \{ |x_1|, |x_2| \} + x_1$, $x = (x_1, x_2)$. Let $V = \{(1, 0) \} \cup \{(1, y) : \frac{1}{2} \leq y \leq 1 \} \cup \{(1, y) : -1 \leq y \leq -\frac{1}{2} \}$. Then $P_f(\theta) = V$, hence $P_f(\theta)$ is not convex. But it is easily seen that $P_f$ is IRL-continuous at $\theta$.

**Corollary 3.7**—If either $V$ is $f$-Chebyshev, or $f$ satisfies property (*) of Proposition 2.1 and $V$ is convex, then $P_f$ is IRL-continuous.

**Proposition 3.8**—Let $f$ be a continuous sublinear function. If either $V$ is a strict $f$-sun or $P_f$ is IRL-continuous, then for each $x$ in $X$,

$$Co(P_f(x)) \subseteq S(x, f_V(x)),$$

where

$$S(x, f_V(x)) = \{ y \in X : f(x - y) = f_V(x) \}.$$

**Proof**: See Govindaraju², Ch. 2.

**Theorem 3.9**—Let $f$ be as in Proposition 2.1 and let $V$ be $f$-proximinal. If either $V$ is a strict $f$-sun or $P_f$ is IRL-continuous, then for each $x$ in $X$

$$Co(P_f(x)) \subseteq S(x, f_V(x)).$$

**Proof**: It suffices to consider $x$ in $X$ such that $f_V(x) = r > 0$. Then $P_{rV}(x) = 1$, $S(x, f_V(x)) = \{ y \in X : p_r(x - y) = 1 \}$. The result is now an easy consequence of Proposition 2.5 (i) of Govindaraju and Pai° and Proposition 3.8.

**Corollary 3.10**—Let $f$ be either a continuous strictly sublinear function or a strictly convex function satisfying hypothesis of Proposition 2.1. Assume that either
\( V \) is a strict \( f \)-sun or \( P_f \) is IRL-continuous and that \( V \) is \( f \)-proximinal, then \( V \) is \( f \)-Chebyshev.

**Corollary 3.11**—Let \( f \) be as in the last corollary and let \( V \) be \( f \)-proximinal. Then \( V \) is \( f \)-Chebyshev if and only if \( P_f \) is IRL-continuous.

**Theorem 3.12**—Let \( f \) be a non-negative, continuous convex function satisfying (i), (ii), (iii) of Theorem 2.3 and (iv) \( f \) is strictly convex and Gâteaux differentiable at each non-zero point of \( X \).

Then, in \( X \), the class of closed convex sets coincides with the class of \( f \)-Chebyshev sets.

**Proof**: This appears in Govindarajulu and Pai\(^{10}\).

**Theorem 3.13**—Let \( f \) be as in the last theorem. Then the following statements are equivalent:

1. \( P_f \) is continuous;
2. \( P_f \) is l.s.c.;
3. \( P_f \) is IRL-continuous;
4. \( V \) is \( f \)-Chebyshev;
5. \( V \) is convex;
6. \( P_f \) is convex-valued;
7. \( V \) is \( f \)-sun.

**Definition 3.14**—Let \( V \) be a closed subset of \( X \) and let \( x_0 \) in \( X \). \( P_f \) is said to be outer radially upper continuous, abbr. ORU-continuous at \( x_0 \), if for each \( v_0 \) in \( P_f (x_0) \) and each open set \( W \) such that \( P_f (x_0) \subseteq W \), there exists a neighbourhood \( U \) of \( x_0 \) in \( \sigma (x_0) \) such that \( P_f (x) \subseteq W \) for every \( x \) in \( U \). \( P_f \) is called ORU-continuous if it is ORU-continuous at each point.

We note that \( P_f \) is ORU-continuous at \( x_0 \) if and only if the restriction of \( P_f \) to \( \sigma (x_0) \) is u.s.c. This enables one to obtain:

**Proposition 3.15**—Consider the following statements:

1. \( P_f \) is ORU-continuous at \( x_0 \);
2. For each \( v_0 \) in \( P_f (x_0) \) and an open neighbourhood \( W \) of origin in \( X \), there exists a neighbourhood \( U \) of \( x_0 \) in \( \sigma (x) \) such that \( P_f (x_0) \subseteq P_f (x_0) + W \) for every \( x \) in \( U \);
(3) Given \( v_0 \) in \( P_f(x_0) \), an open neighbourhood \( W \) of origin in \( X \) and a net \( \{ x_a \} \subseteq \sigma(x_0) \), such that \( x_a \to x_0 \), there exists \( a_0 \) such that \( P_f(x_a) \subseteq P_f(x_0) + U \), for all \( a \geq a_0 \).

(4) For each \( v_0 \) in \( P_f(x_0) \) and each net \( \{ x_a \} \) in \( \sigma(x_0) \) with \( x_a \to x_0 \) and each net \( \{ v_a \} \) with \( v_a \in P_f(x_a) \) and such that \( v_a \to v \), we have \( v \in P_f(x_0) \).

Then \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \). Moreover, if \( P_f(x_0) \) is compact, then the first three statements are equivalent and if \( V \) is compact, then all the four statements are equivalent.

**Theorem 3.16**—Let \( f \) be either a continuous sublinear function or a continuous convex function as in Proposition 2.1 and let \( V \) be \( f \)-proximinal. If \( V \) is an \( f \)-sun and \( P_f \) is compact-valued, then \( P_f \) is ORU-continuous.

**Proof** : Let \( x_0 \) in \( X \) and \( v_0 \) in \( P_f(x_0) \). Let \( x_a = x_0 + d_a(x_a - v_0) \), where \( d_a > 0 \) and \( d_a \to 0 \). Then \( x_0 \to x_a \). Since \( v \) is an \( f \)-sun, we obtain \( P_f(x_0) = \bigcap_{a \geq 0} P_f(x_0) \), as in Proposition 3.3 or Theorem 3.4.

Let \( W \) be an open neighbourhood of origin in \( X \). Since \( \{ P_f(x_a) \} \) is a decreasing net of compact sets, there exists \( a_0 \) such that \( P_f(x_a) \subseteq P_f(x_0) + W \), for \( a \geq a_0 \). By the last proposition, \( P_f \) is ORL-continuous at \( x_0 \).

**Corollary 3.17**—Let \( f \) be as in the last theorem. If \( V \) is an \( f \)-Chebyshev, \( f \)-sun, then \( P_f \) is ORU-continuous.

**References**