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ON SOLAR PROPERTIES OF SETS AND RADIAL CONTINUITY OF SET-VALUED f-PROJECTIONS

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Let V be a nonempty closed subset of a separated locally convex space X. Given a continuous convex function f defined on X, one defines here the so-called f-projection P_f supported on the set V. In this exposition, radial semi-continuity concepts are studied for the set-valued mapping P_f and several relationships between properties of the set V and continuity of the mapping P_f are explored.

1. INTRODUCTION

Let X, Y be a pair of linear spaces put in duality by a separating bilinear form $<,>$ and equipped with locally convex topologies compatible with the pairing. Let f be a continuous convex function defined on X and satisfying $f(\theta) = 0$. Given a nonempty closed subset V of X and x in X, let

$$f_V(x) := \inf \{f(x - v) : v \in V\}$$

and

$$P_f(x) := \{v \in V : f(x - v) = f_V(x)\} \text{ (possibly void).}$$

The set-valued mapping P_f is called f-projection supported on V. V is said to be f-proximinal (resp. f-Chebyshev) if $P_f(x)$ is nonempty (resp. a singleton) for each x in X. In case X is a normed space with the norm topology and f is the given norm, there has been a lot of interest (cf. Vlasov[11] for an excellent survey) in studying properties of the supporting set related to its f-projection (the so-called metric projection in this case). In case f is sublinear, properties of sets related to f-projections have been investigated earlier[7-9]. These have been further extended[10] to the case when f is a

continuous convex function satisfying $f(\theta) = 0$. Brosowski and Deutsch[1-3] have introduced radial continuity concepts for metric projections which are sharper than the usual upper and lower semicontinuity in relation to properties of the supporting set. The principal aim of the present exposition is to extend to f-projections of Pai and Govindarajulu[10], these radial continuity concepts and to investigate properties of the supporting set related to these. Brosowski[4,5] has also extended and applied in a different way radial continuity concepts to the problem of parametric semi-infinite optimization.

2. Preliminary Results and f-suns

Let $f : X \to \mathbf{R}$ be a continuous convex function satisfying $f(\theta) = 0$. For r in $\mathbf{R}$, $r > 0$, let $S_r := \{x \in X : f(x) \leqslant r\}$ denote the sub-level subset of f. S_r is a convex absorbing set containing the origin θ in its interior. Let $P_r(x) := \inf\{\lambda > 0 : x \in \lambda S_r\}$ (x in X) denote the Minkowski gauge of S_r. Then P_r is a non-negative continuous sublinear function.

Recall (cf. Castaing and Valadier[1]) that f is said to be inf-compact if the sub-level sets S_r are compact for each r in $\mathbf{R}$. The set V is said to be an f-sun (resp. a strict f-sun) if for each x in X, v in $P_f(x_a)$ holds for some (resp. each) element v in $P_f(x)$ and each $a \geqslant 1$, where $x_a = v + a(x - v)$.

Proposition 2.1—Let f be a non-negative continuous convex function satisfying $f(x) = 0$ if and only if $x = \theta$ and the property

(*) There exists a continuous bijection $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ ($\mathbf{R}_+$ = non-negative reals) such that $f(ax) = \psi(a) f(x)$ ($a \geqslant 0$ and x in X).

Let V be f-proximinal. If V is an f-sun (resp. a strict f-sun), then V is a P_r-sun (resp. a strict P_r-sun) for each $r > 0$. Conversely, if V is a P_r-sun (resp. a strict P_r-sun) for some $r > 0$, then V is an f-sun (resp. a strict f-sun).

Proof : By Proposition 2.5 (i) of Govindarajulu and Pai[10], $P_f = P_{P_r}$ for each $r > 0$. If v is an f-sun (resp. a strict f-sun), then for each x in X, v in $P_f(x_a) = P_{P_r}(x_a)$ for some (resp. each) element v in $P_f(x) = P_{P_r}(x)$ and each $a \geqslant 1$ and $r > 0$. Hence, V is a P_r-sun (resp. a strict P_r-sun) for each $r > 0$. The proof of the remaining part is identical.

Theorem 2.2—Let P be a continuous, inf-compact sublinear function and let V be p-Chebyshev. Then V is a p-sun.

Proof : See Pai and Govindarajulu[10].

Theorem 2.3—Let f be a non-negative, continuous convex function which satisfies

(i) $f(x) = 0$ if and only if $x = \theta$;

(ii) the property (*) of Proposition 2.1

(iii) f is inf-compact.

Then each f-Chebyshev set is an f-sun.

PROOF : Let V be a closed f-Chebyshev set. In view of (i), it suffices to consider x in X such that $f_V(x) = r > 0$. By Proposition 2.5 (i) of Govindarajulu and Pai[10], $P_f = P_{P_r}$ and V is P_r-Chebyshev. Also, by (iii) P_r is inf-compact and hence, V is P_r-sun by the last theorem. By Proposition 2.1, V is an f-sun.

Recall that V is said to be b-compact Govindarajulu and Pai[7] if for each x in X, there exists b in $\mathbf{R}$ such that $b > f_V(x)$ and $(x - V) \cap S_b$ is compact.

Theorem 2.4—Let f be a non-negative, continuous convex function which satisfies

(i) $f(x) = 0$ if and only if $x = \theta$;

(ii) $f(-x) = f(x)$, x in X;

(iii) the property (*) of Proposition 2.1.

If V is a b-compact, f-Chebyshev set, then V is an f-sun.

PROOF : In view of (i), it suffices to consider x in X such that $f_V(x) = r > 0$. Since V is b-compact, there is $a\ b > r > 0$ such that $(x - V) \cap S_b$ is compact. By property (*), choose $0 < a < 1$ such that $\psi(a) = r/b$.

By Lemma 2.2 of Govindarajulu and Pai[10] $P_b = aP_r$ and $S_b = \{x : P_b(x) \leqslant 1\} = \{x : P_r \leqslant 1/a\}$. Since $1 = \inf\{P_f(x - v) : v \text{ in } V\} < 1/a$, V is b-compact for P_r. Also, by (ii), $P_r(-x) = P_r(x)$, x in X. By Theorem 3.1 of Govindarajulu and Pai[7] V is a P_r-sun and by Proposition 2.1, V is an f-sun.

Theorem 2.5—Let f be as in Theorem 2.3. If V is an f-proximinal set supporting a convex-valued f-projection P_f, then V is an f-sun.

PROOF : It suffices to consider x in X such that $f_V(x) = r > 0$. By Proposition 2.5 (i) of Govindarajulu and Pai[10], $P_f = P_{P_r}$. Now P_r is inf-compact, satisfies $P_r(x) = 0$ if and only if $x = \theta$ and furthermore, P_{P_r} is convex-valued. Therefore, by Theorem 2.6 of Govindarajulu and Pai[8], V is a P_r-sun and by Proposition 2.1, V is an f-sun.

3. Radial Continuity of f-Projection

Definition 3.1—Let V be a closed subset of X and let x_0 in X. P_f is said to be outer radially lower semi-continuous (resp. inner radially lower semi-continuous), abbr. ORL-continuous (resp. IRL-continuous) at x_0 if, for each v_0 in $P_f(x_0)$ and each open set W such that $P_f(x_0) \cap W \neq \phi$, there exists a neighbourhood U of x_0 in the semi-ray $\sigma(x_0) := \{v_0 + a(x_0 - v_0) : a \geq 1\}$ (resp. in the line segment $[v_0, x_0]$) such that $P_f(x) \cap W \neq \phi$ for every x in U. P_f is said to be ORL-continuous (resp. IRL-continuous) if it is ORL-continuous (resp. IRL-continuous) at each point of X.

We note that P_f is ORL-continuous (resp. IRL-continuous) at x_0 if and only if the restriction of P_f to $\sigma(x_0)$ (resp. to $[v_0, x_0]$) is l.s.c. for each v_0 in $P_f(x_0)$. Employing this remark and the known criteria for lower semi-continuity of multi-functions enables one to obtain easily :

Proposition 3.2—The following statements are equivalent

(1) P_f is ORL-continuous (resp. IRL-continuous) at x_0;

(2) For each v_0, v_1 in $P_f(x_0)$ and each neighbourhood W of origin in X, there exists a neighbourhood U of x_0 in $\sigma(x_0)$ (resp. in $[v_0, x_0]$) and an element v in $P_f(x)$ for each x in U such that v_1 in $v + W$;

(3) Given v_0, v_1 in $P_f(x_0)$ and a net $\{x_a\} \subseteq \sigma(x_0)$ (resp. $[v_0, x_0]$) such that $x_a \to x_0$, there exists a subnet $\{x_b\}$ of $\{x_a\}$ and a net $\{v_b\}$ with v_b in $P_f(x_b)$ such that $v_b \to v_1$.

Proposition 3.3—If f is sublinear and V is an f-sun, then P_f is ORL-continuous.

Proof : Let $x_0 \in X$. V being an f-sun, there exists v_0 is $P_f(x_0)$ such that v_0 in $P_f(x_a)$ for every x_a in $\sigma(x_0)$. We assert that $P_f(x_0) = \cap \{P_f(x_a) : a \geq 1\}$, from which the lemma would follow. Indeed let $v_1 \in P_f(x_0)$ and $x_a = v_0 + a(x_0 - v_0)$ for $a > 1$. Then

$$\begin{aligned} f(x_a - v_1) &= f((x_0 - v_1) + (a - 1)(x_0 - v_0)) \\ &\leqslant f(x_0 - v_1) + (a - 1) f(x_0 - v_0) \\ &= af(x_0 - v_0) = f(x_a - v_0), \text{ whence} \\ & v_1 \text{ in } P_f(x_a). \end{aligned}$$

Theorem 3.4—Let f be as in Proposition 2.1 and let V be f-proximinal. If V is an f-sun, then P_f is ORL-continuous.

Proof : It suffices to consider x in X such that $f_V(x) = r > 0$. Then $P_f = P_{P_r}$ and V is a p_r-sun, whence by last lemma P_{P_r} and hence, P_f is ORL-continuous.

Theroem 3.5—Assume that f satisfies property (*) of Proposition 2.1. If $P_f(x_0)$ is convex, then P_f is IRL-continuous at x_0.

PROOF : If $P_f(x_0) = \phi$, the result is trivial. Let v_0, v_1 be in $P_f(x_0)$ and consider $x_a = v_0 + a(x_0 - v_0)$, for $0 < a < 1$. Then $x_a \in [v_0, x_0]$, $v_a = v_0 + a(v_1 - v_0)$ in $P_f(x_0)$ and for $a \to 1$, $v_a \to v_1$. Also,

$$f(x_a - v_a) = f(a(x_0 - v_1)) = \psi(a) f(x_0 - v_1)$$

$$= \psi(a) f(x_0 - v_0) = f(x_a - v_0).$$

Therefore, $v_a \in P_f(x_a)$ and by Proposition 4.2 (3), P_f is IRL-continuous at x_0.

Corollary 3.6—If f satisfies property (*) of Proposition 2.1 and P_f is convex-valued, then P_f is IRL-continuous.

Remark : The converse of Theorem 3.5 is false. For example, let $X = \mathbf{R}^2$, $f(x) = \max\{|x_1|, |x_2|\} + x_1$, $x = (x_1, x_2)$. Let $V = \{(1, 0)\} \cup \{(1, y) : \frac{1}{2} < y \leqslant 1\} \cup \{(1, y) : -1 \leqslant y \leqslant -\frac{1}{2}\}$. Then $P_f(\theta) = V$, hence $P_f(\theta)$ is not convex. But it is easily seen that P_f is IRL-continuous at θ.

Corollary 3.7—If either V is f-Chebyshev, or f satisfies property (*) of Proposition 2.1 and V is convex, then P_f is IRL-continuous.

Proposition 3.8—Let f be a continuous sublinear function. If either V is a strict f-sun or P_f is IRL-continuous, then for each x in X,

$$Co(P_f(x)) \subset S(x, f_V(x)),$$

where

$$S(x, f_V(x)) = \{y \in X : f(x - y) = f_V(x)\}.$$

PROOF : See Govindarajulu[9], Ch. 2.

Theorem 3.9—Let f be as in Proposition 2.1 and let V be f-proximinal. If either V is a strict f-sun or P_f is IRL-continuous, then for each x in X

$$Co(P_f(x)) \subset S(x, f_V(x)).$$

PROOF : It suffices to consider x in X such that $f_V(x) = r > 0$. Then $P_{r,V}(x) = 1$, $S(x, f_V(x)) = \{y \in X : p_r(x - y) = 1\}$. The result is now an easy consequence of Proposition 2.5 (i) of Govindarajulu and Pai[10] and Proposition 3.8.

Corollary 3.10—Let f be either a continuous strictly sublinear function or a strictly convex function satisfying hypothesis of Proposition 2.1. Assume that either

V is a strict f-sun or P_f is IRL-continuous and that V is f-proximinal, then V is f-Chebyshev.

Corollary 3.11—Let f be as in the last corollary and let V be f-proximinal. Then V is f-Chebyshev if and only if P_f is IRL-continuous.

Theorem 3.12—Let f be a non-negative, continuous convex function satisfying (i), (ii), (iii) of Theorem 2.3 and (iv) f is strictly convex and Gâteaux differentiable at each non-zero point of X.

Then, in X, the class of closed convex sets coincides with the class of f-Chebyshev sets.

PROOF : This appears in Govindarajulu and Pai[10].

Theorem 3.13—Let f be as in the last theorem. Then the following statements are equivalent :

(1) P_f is continuous;
(2) P_f is l.s.c.;

(3) P_f is IRL-continuous;

(4) V is f-Chebyshev;

(5) V is convex;

(6) P_f is convex-valued;

(7) V is f-sun.

Definition 3.14—Let V be a closed subset of X and let x_0 in X. P_f is said to be outer radially upper continuous, abbr. ORU-continuous at x_0, if for each v_0 in $P_f(x_0)$ and each open set W such that $P_f(x_0) \subset W$, there exists a neighbourhood U of x_0 in $\sigma(x_0)$ such that $P_f(x) \subset W$ for every x in U. P_f is called ORU-continuous if it is ORU-continuous at each point.

We note that P_f is ORU-continuous at x_0 if and only if the restriction of P_f to $\sigma(x_0)$ is u.s.c. This enables one to obtain :

Proposition 3.15—Consider the following statements :

(1) P_f is ORU-continuous at x_0;

(2) For each v_0 in $P_f(x_0)$ and an open neighbourhood W of origin in X, there exists a neighbourhood U of x_0 in $\sigma(x)$ such that $P_f(x_0) \subset P_f(x_0) + W$, for every x in U;

(3) Given v_0 in $P_f(x_0)$, an open neighbourhood W of origin in X and a net $\{x_a\} \subseteq \sigma(x_0)$, such that $x_a \to x_0$, there exists a_0 such that $P_f(x_a) \subset P_f(x_0) + U$, for all $a \geqslant a_0$.

(4) For each v_0 in $P_f(x_0)$ and each net $\{x_a\}$ in $\sigma(x_0)$ with $x_a \to x_0$ and each net $\{v_a\}$ with v_a in $P_f(x_a)$ and such that $v_a \to v$, we have v in $P_f(x_0)$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, if $P_f(x_0)$ is compact, then the first three statements are equivalent and if V is compact, then all the four statements are equivalent.

Theorem 3.16—Let f be either a continuous sublinear function or a continuous convex function as in Proposition 2.1 and let V be f-proximinal. If V is an f-sun and P_f is compact-valued, then P_f is ORU-continuous.

PROOF : Let x_0 in X and v_0 in $P_f(x_0)$. Let $x_a = x_0 + d_a(x_0 - v_0)$, where $d_a > 0$ and $d_a \to 0$. Then $x_0 \to x_a$. Since v is an f-sun, we obtain $P_f(x_0) = \bigcap_{a>0} P_f(x_a)$, as in Proposition 3.3 or Theorem 3.4.

Let W be an open neighbourhood of origin in X. Since $\{P_f(x_a)\}$ is a decreasing net of compact sets, there exists a_0 such that $P_f(x_a) \subset P_f(x_0) + W$, for $a \geqslant a_0$. By the last proposition, P_f is ORL-continuous at x_0.

Corollary 3.17—Let f be as in the last theorem. If V is an f-Chebyshev, f-sun, then P_f is ORU-continuous.

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