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ON THE DISTRIBUTION OF VALUES OF DIFFERENTIAL POLYNOMIALS

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If f is a meromorphic function satisfying $\bar{N}(r, f) + \bar{N}(r, 1/f) = S(r, f)$ and P is a homogeneous differential polynomial in f which does not reduce to a constant, it is shown that the order of P equals the order of f and $\bar{N}(r, P) + \bar{N}\left(r, \frac{1}{P}\right) = S(r, P)$ so that $\Theta(a, P) = 0$ for all a, $a \neq 0, \infty$.

The impossibility of certain identities between meromorphic functions is deduced improving earlier results of Hiong [*C. R. Acad. Sci.* (1956), 222-25].

We denote by C the set of all finite complex numbers and by $\bar{C}$ the extended complex plane consisting of all (finite)c omplex numbers and ∞. By a meromorphic function we shall always mean a transcendental meromorphic function in the plane. We use the usual notations of the Nevanlinna theory of meromorphic functions as explained in Hayman[2] and Singh and Gopalakrishna[5].

If f is a meromorphic function we denote by $S(r, f)$ any quantity satisfying

$$\int_{r_0}^{r} \frac{S(x, f)}{x^{1+\lambda}}\, dx = O\left(\int_{r_0}^{r} \frac{\log T(x, f)}{x^{1+\lambda}}\, dx\right) \qquad \ldots(1)$$

as $r \to \infty$, whenever $\lambda > 0$ and

$$S(r, f) = o(T(r, f)) \qquad \ldots(2)$$

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as $r \to \infty$, through all values if f is of finite order and outside a set of finite linear measure if f is of infinite order.

If f is a meromorphic function, then we have the following fundamental results of Nevanlinna[4] (p. 63).

$$m(r, f'/f) = S(r, f)$$

and

$$(q-2)\,T(r,f) \leqslant \sum_{i=1}^{q} N(r, a_i, f) - N_1(r) + S(r,f) \qquad \ldots(3)$$

whenever $a_1, a_2, \ldots, a_q$ are distinct elements of $\bar{C}$, where

$$N_1(r) = 2\,N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right).$$

Generalisations and extensions of these results have been obtained by Milloux, Hayman and others and most of them are found in Hayman[2]. Hayman[2] denotes by $S(r,f)$ any quantity satisfying (2) above. However, since all the results are obtained from the fundamental results of Nevanlinna it is easy to see that the theorems in Hayman[2] are valid with $S(r,f)$ satisfying (1) and (2) also.

In particular, we have (Hayman[2], Theorem 3.1), for a meromorphic function f,

$$m(r, f^{(k)}/f) = (S\,r, f) \qquad \ldots(4)$$

for $k \geqslant 1$.

If f is a meromorphic function of order ρ, $0 \leqslant \rho \leqslant \infty$ and $a \in \bar{C}$, we define

$$\bar{\rho}\,(a,f) = \limsup_{r\to\infty} \frac{\log^+ \bar{n}(r,a,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^+ \bar{N}(r,a,f)}{\log r}$$

and we call a an *evB* (exceptional value in the sense of Borel) for f for distinct zeros if $\bar{\rho}\,(a,f) < \rho$.

In this paper we shall be concerned with meromorphic functions P which are polynomials in the meromorphic function f and the successive derivatives of f with coefficients $a(z)$ where the $a(z)$ are meromorphic functions satisfying

$$T(r, a(z)) = S(r,f).$$

As in Hayman[2] such functions will be called differential polynomials in f. Thus a differential polynomial P in f is the sum of a finite number of terms of the form

$af^{l_0}(f^{(1)})^{l_1}\dots(f^{(k)})^{l_k}$ where $l_0, l_1,\dots l_k$ are non-negative integers. The integer $l_0+l_1+\dots+l_k$ is called the degree of this term. As usual, the degree of P is the highest of the degrees of its terms and P is called homogeneous if all its terms have the same degree.

Yang[6] proved the following theorem.

Theorem A—Let f be a meromorphic function with

$$N(r,f)+N(r,1/f)=S(r,f)$$

and P be a differential polynomial in f of the form

$$P=\Sigma a f^{l_0}(f^{(1)})^{l_1}\dots(f^{(k)})^{l_k}$$

with no constant term. Furthermore assume that the degree n of P is greater than one and $l_0<n$, $0\leqslant l_i\leqslant n$ for all $i\neq 0$. Then $\delta(a,P)<1$ for all $a\neq 0,\infty$.

In this paper we obtain a stronger result if P is a homogeneous differential polynomial in f. Our result also generalizes, to a very great extent, the following theorem of Hiong[3].

Theorem B—Let $F(z)$ and $G(z)$ be entire functions having no zeros. Let $a_i(z)$ $(i=1,2,\dots,k)$ be entire functions whose orders are less than the order of $F(z)$. Then the identity

$$\sum_{i=1}^{k} a_i(z)F^{(i)}(z)+G(z)\equiv 1$$

is impossible.

Remark: If f is a meromorphic function and if $\bar{N}(r,a,f)=S(r,f)$ and $\bar{N}(r,b,f)=S(r,f)$ for distinct elements a, b in $\bar{C}$, it follows from (3) that $\bar{N}(r,c,f)\sim T(r,f)$ as r tends to ∞ outside a set of finite measure so that $\Theta(c,f)=0$ for all $c\in\bar{C}$, $c\neq a,b$. It also follows from (3) that no element of $\bar{C}-\{a,b\}$ is an *ev*B for f for distinct zeros, for, if c where such an element then we can choose positive numbers μ,λ such that $\mu<\lambda<$ order of f and $\bar{N}(r,c,f)=O(r^{\mu})$.

Then

$$\begin{aligned}T(r,f)&<\bar{N}(r,a,f)+\bar{N}(r,b,f)+\bar{N}(r,c,f)+S(r,f)\\&=\bar{N}(r,c,f)+S(r,f)\end{aligned}$$

which implies, in view of (1), that

$$\int_{r_0}^{\infty} \frac{T(x,f)}{x^{1+\lambda}}\,dx < \infty.$$

Hence, the order of f would not exceed λ which would be a contradiction.

If f is of finite order, then, by (2), $\bar{N}(r, a, f) = S(r, f)$ where $a \in \bar{C}$ implies that $\Theta(a, f) = 1$.

Theorem 1—Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}(r, 1/f) = S(r, f). \qquad \ldots(5)$$

If P is a homogeneous differential polynomial in f which does not reduce to a constant, then the order of P equals the order of f and

$$\bar{N}(r, P) + \bar{N}\left(r, \frac{1}{P}\right) = S(r, P) \qquad \ldots(6)$$

so that $\bar{N}\left(r, \frac{1}{P-a}\right) \neq S(r, P)$ and $\Theta(a, P) = 0$ for all $a \in \bar{C} - \{0, \infty\}$. Also, no element, of $\bar{C} - \{0, \infty\}$ is an *evB* for P for distinct zeros.

PROOF : We have (Gopalakrishna and Bhoosnurmath[1], Lemma 1)

$$m(r, P/f^n) = S(r, f). \qquad \ldots(7)$$

If $f^{(m)}$ is the highest derivative of f occurring in P then it is easily seen that a zero or pole of f, which is not a zero or pole of anycoeffi cient $a(z)$ of P, is a pole of P/f^n of order at most mn. Hence,

$$N(r, P/f^n) \leqslant mn\{\bar{N}(r, f) + \bar{N}(r, 1/f)\} + S(r, f). \qquad \ldots(8)$$

(We ignore, here and in the sequel, the poles and the zeros of coefficients $a(z)$ of P since $N(r, a(z)) + N\left(r, \frac{1}{a(z)}\right) \leqslant 2T(r, a(z)) + O(1) = S(r, f)$ for each coefficient).

Now,

$$\begin{aligned} n\,T(r, f) &= T(r, f^n) \\ &\leqslant T(r, f^n/P) + T(r, P) \\ &= T(r, P/f^n) + T(r, P) + O(1) \\ &\leqslant T(r, P) + mn\{\bar{N}(r, f) + \bar{N}(r, 1/f)\} + S(r, f) \end{aligned}$$

by (7) and (8).

Thus, by (5),

$$n\,T(r,f) \leqslant T(r,\,P) + S(r,f) \qquad \ldots(9)$$

whence it follows that

$$S(r,f) = S(r,\,P). \qquad \ldots(10)$$

On the other hand,

$$\begin{aligned} T(r,\,P) &\leqslant T(r,\,P/f^n) + T(r,f^n) \\ &\leqslant n\,T(r,f) + S(r,f) \qquad \ldots(11) \end{aligned}$$

by (7), (8) and (5).

From (9) and (11) it follows that the orders of f and P are equal.

Also,

$$\begin{aligned} \bar{N}(r,\,P) &\leqslant \bar{N}(r,f) + S(r,f) \\ &= S(r,f) \text{ by (5)} \\ &= S(r,\,P) \text{ by (10)} \end{aligned}$$

and

$$\begin{aligned} \bar{N}\left(r,\,\frac{1}{P}\right) &\leqslant \bar{N}(r,f^n/P) + \bar{N}(r,\,1/f^n) \\ &\leqslant \bar{N}(r,\,1/f) + T(r,f^n/P) \\ &= \bar{N}(r,\,1/f) + T(r,\,P/f^n) + O(1) \\ &= S(r,f), \text{ by (7), (8) and (5)} \\ &= S(r,\,P) \text{ by (10).} \end{aligned}$$

This completes the proof of Theorem 1.

Remark : In Theorem 1, if f is of finite order then (5) implies that $\Theta(0,f) = \Theta(\infty,f) = 1$ and it follows that $\Theta(0,\,P) = \Theta(\infty,\,P) = 1$.

Corollary 1.1—Let f be a meromorphic function satisfying (5). Let P be a homogeneous differential polynomial in f which does not reduce to a constant. If G is a meromorphic function such that

$\Theta(a, G) > 0$ or $\bar{N}\left(r, \frac{1}{G-a}\right) = S(r, G)$ for some $a \in C$, $a \neq 1$, then the identity

$$P + G \equiv 1 \qquad \ldots(12)$$

is impossible.

PROOF : By Theorem 1, we have

$$\bar{N}\left(\frac{1}{P-(1-a)}\right) \neq S(r, P) \text{ and } \Theta(1-a, P) = 0.$$

Suppose (12) holds. Then $T(r, P) \sim T(r, G)$

and

$$\bar{N}\left(r, \frac{1}{P-(1-a)}\right) = \bar{N}\left(r, \frac{1}{G-a}\right).$$

Hence

$$\bar{N}\left(r, \frac{1}{P-(1-a)}\right) = S(r, G) = S(r, P)$$

or

$$\Theta(1-a, P) = \Theta(a, G) > 0,$$

which is a contradiction. This proves the corollary. Corollary 1.1 includes Theorem *B* as a very special case.

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