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ON TORSION OF RECTANGULAR CYLINDERS WITH GROOVES

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The torsion problem of a rectangular cylindrical beam under a terminal couple with triangular, circular, elliptic and parabolic grooves has been studied. The shearing stresses, the lines of shearing stress and the distorted cross-sections of the beam have been obtained by Saint-Venant's semi-inverse method. The mininum shearing stress occurs at the vertices of the grooves. The contour lines are straight lines. The lines of shearing stresses take the shapes of the grooves in the beam but in case of a triangular groove, they are hyperbolic curves.

1. INTRODUCTION

Twisting and slotting of cylindrical and prismatic bars of various cross-sections is of great practical utility and technical importance in mechanical engineering and structural design. It has its application in railway construction, steel bridges, huge buildings, ship building, machine construction, defence equipment, carriage and vehicle manufacturing etc.

For the first time, the torsion problem of a cylindrical beam with grooves was considered by Weber[1]. He has studied the effect on maximum shearing stress of a circular groove in a circular shaft. Recently Dooley[2] has studied the problem of torsion of non-circular cylinders by a simplified method and has obtained an approximate value for the maximum shearing stress. The problem of torsion of a parabolic cylinder under a terminal couple has been considered by Deshwal[3] and the terminal couple has been expressed in terms of the polar moment of interia. Tsuji and Shibuya[4] have investigated the torsion problem of a cylindrical rod composed of a semi-infinite solid and hollow cylinders. A stress singularity appears at the corner of the connected rod and the method of series expansion in consideration of the stress singularity at the corner is applied. The problem of torsion of a rectangular beam with circular and elliptic ends has been studied by Deshwal and Singh[5]. The contour lines are rectangular hyperbolae and the maximum shearing stress occurs at the mid-points of the beam.

The torsion problem of an elastic solid cylinder with one or two semi-circular grooves is discussed by Hasegawa and Kuriyama[6]. Their results are in agreement with Nisitani's results[12] calculated by the body force method and Kato's experimental results[13].

We propose to discuss here the problem of twisting and slotting of a rectangular cylindrical beam with triangular, circular, elliptic, hyperbolic and parabolic grooves at one end. One end of the beam is fixed and the other is twisted by a couple of moment M directed along the axis of the beam. The couple produces stresses in each section of the beam. Let C be a boundary of a cross-section of the beam and R be the region inside C. The axis of the beam is along the z-axis of the co-ordinate system and the cross-section is in the xy-plane. The method of solution is the semi-inverse method due to Saint-Venant[7]. A function $\psi(x, y)$ is to be determined such that it satisfies the equation

$$\nabla^2 \psi = 0 \qquad \ldots(1)$$

in the region R and on C, it has the value

$$\psi(x, y) = (x^2 + y^2)/2. \qquad \ldots(2)$$

2. A Triangular Groove

We introduce a function

$$\bar{\phi}(x, y) = \psi(x, y) + \frac{c}{2}(y^2 - x^2) + (c - 1)ax - \frac{c - 1}{2}a^2 \qquad \ldots(3)$$

where a and c are constants and $\bar{\phi}(x, y)$ is harmonic. From (2) and (3), we have

$$\bar{\phi}(x, y) = \frac{1}{1 - m^2}[y^2 - m^2(x - a)^2], \quad c = \frac{1 + m^2}{1 - m^2} \qquad \ldots(4)$$

$$= 0, \text{ on the lines } y = \pm m(x - a), \qquad \ldots(5)$$

$$= \frac{1}{1 - m^2}[b^2 - m^2(x - a)^2], \; y = \pm b. \qquad \ldots(6)$$

The general solution of (1) is

$$\bar{\phi}(x, y) = \Sigma A_n \sin c_n x \cosh c_n y \qquad \ldots(7)$$

where A_n and c_n are constants to be determined from the boundary conditions. Using (5), we have

$$\sin c_n a = 0 \text{ on the point } (a, 0) \text{ on } C$$

or

$$c_n = n\pi/a,\ n = 0, 1, 2, 3,\ldots \qquad \ldots(8)$$

Using (6), we find

$$\frac{1}{1 - m^2}\left[b^2 - m^2 (x - a)^2\right] = \Sigma A_n \sin c_n x \cosh c_n b. \qquad \ldots(9)$$

Multiplying both sides by $\sin c_m x$ and integrating between $x = -d$ to $x = d$ ($d = pa$, p is a positive integer $\geqslant 2$), we obtain

$$A_n = -\frac{4am^2}{1 - m^2} \cdot \frac{\cos c_n d}{c_n \cosh c_n b} \qquad \ldots(10)$$

where the following integrals have been used

$$\int_{-d}^{d} \sin c_n x \sin c_m x\, dx = d, \text{ if } m = n$$

$$= 0, \text{ if } m \neq n. \qquad \ldots(11)$$

Thus the general solution (7) is now

$$\bar{\phi}\ (x, y) = -\frac{4am^2}{1 - m^2} \sum_n \frac{\cos c_n d \sin c_n x \cosh c_n y}{c_n \cosh c_n b} \qquad \ldots(12)$$

and

$$\psi\ (x, y) = \bar{\phi}\ (x, y) - \frac{1 + m^2}{2\,(1 - m^2)}\,(y^2 - x^2) - \frac{2\,m^2\,ax}{1 - m^2} + \frac{m^2\,a^2}{1 - m^2}\,. \qquad \ldots(13)$$

Since $n = 0$, corresponds to a trivial solution in (12) and so $n \neq 0$ further $m \neq \pm 1$, as it makes c infinite. The shearing stresses are given by Sokolnikoff[8] (p. 121).

$$p_{zx} = \mu\alpha\,[\partial\psi/\partial y - y],\ p_{zy} = \mu\alpha\,[-\partial\psi/\partial x + x] \qquad \ldots(14)$$

where μ is the shear modulus and α is the twist per unit length. Using the value of ψ in (14), we have

$$p_{zx} = \frac{\mu\alpha}{1 - m^2}\left[-4am^2 \sum_1^{\infty} \frac{\cos c_n d \sin c_n x \sinh c_n y}{\cosh c_n b} - 2y\right] \qquad \ldots(15)$$

$$p_{zy} = \frac{\mu\alpha}{1 - m^2}\left[4\, am^2 \sum_{1}^{\infty} \frac{\cos c_n d \cos c_n x \cosh c_n y}{\cosh c_n b} - 2m^2 x + 2m^2 a\right]. \qquad \ldots(16)$$

It can be seen that $p_{zx} = 0$ on the x-axis ($y = 0$) such that the component of the shearing stress directed along the central line (x-axis) is zero on the central line. The minimum shearing stress occurs at the vertex $(a, 0)$ of the triangular groove and is found to be

$$p_{\min} = \frac{4\, am^2\, \mu\alpha}{1 - m^2} \sum_{1}^{\infty} \frac{\cos c_n d \cos c_n a}{\cosh c_n b}, \; c_n = \frac{n\pi}{a} \qquad \ldots(17)$$

where

$$p^2 = p_{zx}^2 + p_{zy}^2. \qquad \ldots(18)$$

The infinite series in (17)

$$\sum_{n=1}^{\infty} \frac{\cos c_n d \cos c_n a}{\cosh c_n b} = \sum_{1}^{\infty} \frac{(-1)^n \cos c_n d}{\cosh c_n b} \leqslant \sum_{1}^{\infty} \operatorname{sech} c_n b$$

$$= 2\sum_{1}^{\infty} \frac{e^{-n\pi b/a}}{1 + e^{-2n\pi b/a}} < 2\sum_{1}^{\infty} e^{-n\pi b/a}$$

$$= 2\, e^{-\pi b/a}/1 - e^{-\pi b/a} = \qquad \ldots(19)$$

is convergent. The minimum shearing stress is

$$p_{\min} \doteq 4a\, \mu\alpha\bar{a}\, m^2/(1 - m^2). \qquad \ldots(20)$$

At the corners $(d, \pm b)$ of the beam, the stresses are

$$p_{zx} = \pm\, 2\, \mu b\alpha/(1 - m^2),\; p_{zy} = \frac{2\, \mu\alpha m^2}{1 - m^2}\,[a - d + 2a\, \Sigma \cos^2 c_n d]. \qquad \ldots(21)$$

The infinite series in p_{zy} is not convergent and there is a stress singularity at the corners of the beam.

The torsinal moment is given by

$$M = 4\int_0^d\int_0^b (x\,p_{zy} - y p_{zx})\,dx\,dy - 2\int_a^d\int_0^{m(x-a)} (x p_{zy} - y p_{zx})\,dx\,dy$$

$$= \frac{2\,\mu\alpha}{1-m^2}\left[2am^2\sum\frac{\cos c_n d}{\cosh c_n b}\left[\frac{pm\,(-1)^{np}}{n\pi\,(1+m^2)}\cosh m\pi\,(p-1)\right.\right.$$

$$+ \frac{(1-m^2)\,a^2\,(-1)^{np}}{n^2\,\pi^2\,(1+m^2)^2}\sinh m\pi\,(p-1) - \frac{m\,(-1)^n}{n\pi\,(1+m^2)}$$

$$+ \frac{(-1)^{np}\,ma^3}{n^2\,\pi^2\,(1+m^2)}\left(-(p-1)+\frac{2m}{n\pi\,(1+m^2)}\right)\sinh mn\pi\,(p-1)$$

$$+ \frac{(-1)^{np}\,ma^3}{n^3\,\pi^3\,(1+m^2)}\cosh nm\pi\,(p-1) - \frac{(-1)^n\,a^3}{n^3\,\pi^3\,(1+m^2)}\Big]$$

$$\left. - \frac{m^3\,(d-a)^3\,(d+a)}{6}\right]. \qquad \ldots(22)$$

The infinite series

$$\sum_1^\infty (-1)^{np}\,\frac{\cos c_n d}{n\cosh c_n b} < \sum_1^\infty \operatorname{sech}(n\pi b/a) \doteq \frac{2e^{-\pi b/a}}{1-e^{-\pi b/a}} = \bar{a} \qquad \ldots(23)$$

is convergent. Similarly

$$\frac{(-1)^{np}\cos c_n d}{n^2\cosh c_n b} \doteq \bar{a}, \quad \sum\frac{(-1)^n\cos c_n d}{n^3\cosh c_n b} \doteq \bar{a}. \qquad \ldots(24)$$

If $b = m\,(d-a)$, then

$$\sum\frac{(-1)^{np}\cos c_n d\cosh nm\pi\,(p-1)}{n^3\cosh c_n b} = \sum\frac{(-1)^{np}\cos c_n d}{n^3} = \bar{b} \qquad \ldots(25)$$

is a convergent series[9]. And the series

$$\sum\frac{(-1)^{np}\cos c_n d\sinh mn\pi\,(p-1)}{n^2\cosh c_n b}$$

$$= \sum\frac{(-1)^{np}\cos c_n d\tanh mn\pi\,(p-1)}{n^2}$$

$$< \sum\frac{1}{n^2} = \frac{\pi^2}{6} \qquad \ldots(26)$$

too is convergent. Thus all the series in (22) are convergent and it is obtained that

$$M_{max} \doteq \frac{2\mu\alpha}{\pi(1-m^4)}\left[2am^2\left[pm\bar{a}\cosh m\pi(p-1)\right.\right.$$

$$+\frac{(1-m^2)a^2\bar{a}\sinh m\pi(p-1)}{\pi(1+m^2)}$$

$$\left.-m\bar{a}+\frac{a^3\bar{b}}{\pi^2}-\frac{a^3\bar{a}}{\pi^2}+\frac{2m^2a^3\bar{b}}{\pi^2(1+m^2)}-\frac{(p-1)m\pi a^3}{6}\right]$$

$$\left.-\frac{\pi b^3(d+a)(1+m^2)}{6}\right]. \qquad \ldots(27)$$

The torsion function, apart from a constant, is given by

$$\phi(x,y)=\int_{(0,0)}^{(x,y)}\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy\right)=\int_{(0,0)}^{(x,y)}\left(\frac{\partial\psi}{\partial y}dx-\frac{\partial\psi}{\partial x}dy\right)$$

$$=\frac{4am^2}{1-m^2}\sum\frac{\cos c_nd\cos c_nx\sinh c_ny}{c_n\cosh c_nb}-\frac{1+m^2}{1-m^2}xy$$

$$+\frac{2am^2}{1-m^2}y \qquad \ldots(28)$$

ϕ and ψ are connected by Cauchy-Riemann equations and the contour lines are obtained by $\phi(x, y) = \text{const.}$ Expanding $\cos c_nx$ and $\sinh c_ny$ to small x and y, it is found that

$$y(x-2am^2(2\bar{a}+1)/(1+m^2)) = \text{const.} \qquad \ldots(29)$$

These are rectangular hyperbolae with centre at $\left(\frac{2am^2(2\bar{a}+1)}{(1+m^2)}, 0\right)$. The lines of shearing stress are obtained from

$$\psi-(x^2+y^2)/2 = \text{const.} \qquad \ldots(30)$$

These are hyperbolae

$$m^2(x-a(2\bar{a}+1))^2-y^2 = \text{const.} \qquad \ldots(31)$$

with centre at $(2a\bar{a}+a, 0)$.

3. A Circular Groove

For a circular groove, we take

$$\bar{\phi}(x,y)=\psi(x,y)-xd+ad-a^2/2. \qquad \ldots(32)$$

On C, we have

$$\bar{\phi}\ (x, y) = \frac{1}{2}\ (x^2 + y^2) - xd + ad - a^2/2$$

$$= \frac{1}{2}\ [(x - d)^2 + y^2 - (d - a)^2]$$

$$= 0, \text{ on the circle } (x - d)^2 + y^2 = (d - a)^2 \qquad \ldots(33)$$

$$= \frac{1}{2}\ [(x - d)^2 + b^2 - (d - a^2)],\ y = \pm b. \qquad \ldots(34)$$

Because of the boundary conditions (33) and (34), we find

$$\bar{\phi}\ (x, y) = \sum \frac{2d \cos c_n d}{c_n \cosh c_n b}\ \sin c_n x \cosh c_n y,\ c_n = n\pi/a. \qquad \ldots(35)$$

The stress function $\psi\,(x, y)$ is given by

$$\psi\ (x, y) = \bar{\phi}\ (x, y) + xd - ad + a^2/2 \qquad \ldots(36)$$

and the shearing stresses are found to be

$$p_{zx} = \mu\alpha \left[2d \sum \frac{\cos c_n d}{\cosh c_n b} \cdot \sin c_n x \sinh c_n y - y \right] \qquad \ldots(37)$$

$$p_{zy} = \mu\alpha \left[-2d \sum \frac{\cos c_n d}{\cosh c_n b} \cos c_n x \cosh c_n y + x - d \right]. \qquad \ldots(38)$$

The minimum shearing stress is at the point $(a, 0)$ and is of the magnitude

$$p_{\min} \doteq \mu\alpha\ (d - a + 2\bar{a}d). \qquad \ldots(39)$$

P_{zx} is zero on the line $y = 0$ and there is a stress singularity at the corners of the beam. The torsion function $\phi\ (x, y)$ has the value

$$\phi\ (x, y) = 2d \sum \frac{\cos c_n d\ \cos c_n x \sinh c_n y}{c_n \cosh c_n b} - yd. \qquad \ldots(40)$$

Setting $\phi\ (x, y) = \text{const.}$, the distorted cross-sections for small values of x and y are given by

$$y\ (2\bar{a} - 1) = \text{const.} \qquad \ldots(41)$$

These are straight lines parallel to the x-axis. And the lines of shearing stress

$$\psi\ (x, y) - (x^2 + y^2)/2 = \text{const.} \qquad \ldots(42)$$

are obtained to be the circles

$$(x - d(2a + 1))^2 + y^2 = \text{const.} \qquad \ldots(43)$$

with centre at $(d(2a + 1), 0)$.

4. An Elliptic Groove

We choose

$$\bar{\phi}(x, y) = \psi(x, y) - \frac{c}{2}(x^2 - y^2) - (1 - c)dx - \frac{1 - c}{2}a^2 + (1-c)ad \qquad \ldots(44)$$

and on C, it is found to be

$$\bar{\phi}(x, y) = \frac{1 - c}{2}(x - d)^2 + \frac{1 + c}{2}y^2 - \frac{1 - c}{2}(d - a)^2$$

$$= 0, \text{ on the ellipse } \frac{(x - d)^2}{(d - a)^2} + \frac{y^2}{\frac{(1 - c)}{1 + c}(d - a)^2} = 1$$

$$= \frac{1}{2}[(1 - c)(x - d)^2 + (1 + c)b^2 - (1 - c)(d - a)^2],$$

$$y = \pm b. \qquad \ldots(45)$$

And the solution

$$\bar{\phi}(x, y) = \Sigma A_n \sin c_n x \cosh c_n y, \qquad \ldots(46)$$

due to the boundary conditions (45) is

$$\bar{\phi}(x, y) = 2(1 - c)d \sum \frac{\cos c_n d \sin c_n x \cosh c_n y}{c_n \cosh c_n b}. \qquad \ldots(47)$$

The constant $c = (A^2 - B^2)/(A^2 + B^2)$ where A and B are the lengths of semi major and minor axes of the ellipse. The stress function is

$$\psi(x, y) = \bar{\phi}(x, y) + \frac{c}{2}(x^2 - y^2) + (1 - c)xd + \frac{1 - c}{2}a^2 - (1-c)ad. \qquad \ldots(48)$$

In this case, the shearing stresses are

$$p_{zx} = \mu\alpha\left[2(1 - c)d \sum \frac{\cos c_n d \sin c_n x \sinh c_n y}{\cosh c_n b} - cy - y\right] \qquad \ldots(49)$$

$$p_{zy} = \mu\alpha\Big[-2(1-c)\,d\sum\frac{\cos c_n d\cos c_n x\cosh c_n y}{\cosh c_n b} - cx + x - (1-c)\,d\Big]. \qquad ...(50)$$

The minimum stress occurs at the vertex of the ellipse and is obtained to be

$$p_{\min} \doteq \mu\alpha\,(1-c)\,(2\bar{a}d + d - a). \qquad ...(51)$$

The contour lines are

$$-2(1-c)\,d\sum\frac{\cos c_n d\cos c_n x\sinh c_n y}{c_n\cosh c_n b} - cxy - (1-c)\,yd = \text{const.} \qquad ...(52)$$

These are the same rectangular hyperbolae as obtained in (28) for a triangular groove if c is replaced by $(1+m^2)/(1-m^2)$ and d by a. The lines of shearing stress are the ellipses

$$(x - d\,(2\bar{a}+1))^2 + y^2/(1-c)/(1+c) = \text{constant.} \qquad ...(53)$$

with centre at the point $(d\,(2\bar{a}+1), 0)$. The groove is an ellipse if $c < 1$, a circle if $c = 0$ and a hyperbola if $c > 1$. The results for a circular and hyperbolic groove are particular cases of an elliptic groove if $c = 0$ and $c > 1$.

5. A Parabolic Groove

Here

$$\bar{\phi}(x, y) = \psi(x, y) + \frac{1}{2}(y^2 - x^2) - 4\bar{c}\,(x - a). \qquad ...(54)$$

The boundary C of the cross-section is given by

$$\bar{\phi}(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(y^2 - x^2) - 4\bar{c}\,(x - a)$$

$$= y^2 - 4\bar{c}\,(x - a)$$

$$= 0, \text{ on the parabola } y^2 = 4\bar{c}\,(x - a) \qquad ..(55)$$

$$= b^2 - 4\bar{c}\,(x - a),\ y \pm b. \qquad ...(56)$$

$4\bar{c}$ is the length of the latus-rectum. The solution is obtained to be

$$\bar{\phi}(x, y) = 8\bar{c} \sum \frac{\cos c_n d \sin c_n x \cosh c_n y}{c_n \cosh c_n b} \qquad ...(57)$$

and the stress function is given by

$$\psi(x, y) = \bar{\phi}(x, y) - (y^2 - x^2)/2 + 4\bar{c}(x - a). \qquad ...(58)$$

The shear stresses are

$$p_{zx} = \mu\alpha \left[8\bar{c} \sum \frac{\cos c_n d \sin c_n x \sinh c_n y}{\cosh c_n b} - 2y \right] \qquad ...(59)$$

$$p_{zy} = \mu\alpha \left[-8\bar{c} \sum \frac{\cos c_n d \cos c_n x \cosh c_n y}{\cosh c_n b} - 4\bar{c} \right] \qquad ...(60)$$

and

$$p_{\min} \doteq 4\mu\alpha\bar{c}(2\bar{a} + 1). \qquad ...(61)$$

The distorted cross-sections are given by

$$y(x + 4\bar{c}(2\bar{a} + 1)) = \text{const.} \qquad ...(62)$$

which are rectangular hyperbolae with their centres as $(-4\bar{c}(2\bar{a} + 1), 0)$. The lines of shearing stress are the parabolas

$$y^2 = 4\bar{c}(2\bar{a} + 1)(x - \text{const.}) \qquad ...(63)$$

with their latus-rectum equal to $4\bar{c}(2\bar{a} + 1)$.

6. Conclusions

Grooves are elliptic or hyperbolic according as $c \lesssim 1$ and circular if $c = 0$. The results for circular and hyperbolic grooves are particular cases of those of the elliptic groove. The contour lines for triangular and parabolic grooves can be deduced from those of the elliptic groove by suitably replacing the constants. The contour lines for a circular groove are the dotted straight lines in Fig. 2. For triangular, elliptic and parabolic grooves, the distorted cross-sections are concentric rectangular hyperbolae shown by dotted curves in Figs. 1, 3, 4. The lines of shearing stress take the shapes of their respective grooves (dotted curves in the figures) barring a triangular groove in that case the lines of shearing stress are hyperbolic curves.

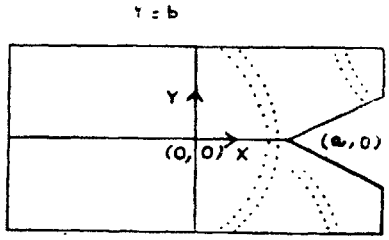

FIG. 1. A trianagular groove.

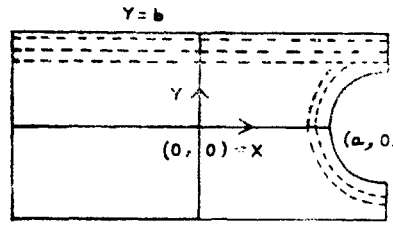

FIG. 2. A circular groove.

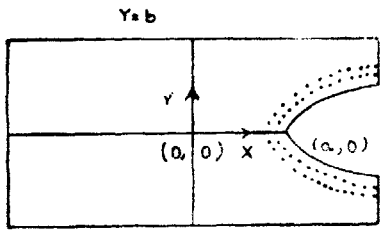

FIG. 3. An elliptic groove.

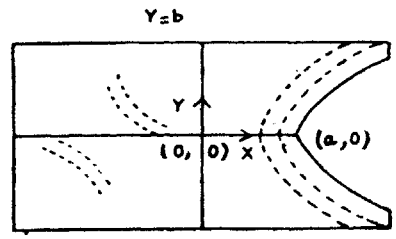

FIG. 4. A parabolic groove.

The component of the shearing stress along the central line of the cross-section of the beam is absent on the central line. The stress is minimum at the vertices of the grooves on the central line and a stress singularity occurs at the corners of the beam. The results for a triangular groove are valid only when $m < 1$ i.e. when the sides of the triangular beam are included at an angle less than 45° with the central line. Tsuji and Shibuya[4] have shown that a stress singularity occurs at the conrners of a rod connected with hollow cylinders. Prescott[10] (p. 148), Goodier[11] (p. 300) and Sokolnikoff[8] (p. 125, 131) have pointed out the stress is minimum at the corners of rectangular and triangular beams.

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