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ON DISTRIBUTIONAL MATHIEU TRANSFORMS

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The aim of present work is to study the expansions of distributions in terms of the Mathieu functions and Mathieu transforms, and the extension of some results of Sharma and Gupta [*Bull. Calcutta Math. Soc.* **72** (1980), 125-33], to a certain class of generalized functions. Finally, the transform developed in this paper has been applied to the problem of heat conduction between two parallel plates where the source of heat generation depends upon temperature linearly.

1. INTRODUCTION

Zemanian[6] has developed the orthogonal series expansion of generalized functions and generalized integral transformations. Expansions of generalized functions with respect to some general classes of complete orthonormal system are investigated by Pandey and Pathak[4]. A recent survey on distributional transforms can be found in the paper of Koh[2]. This paper is devoted to the study of expansions of distributions in terms of the generalized functions and Mathieu transforms. We shall define the Mathieu transform of f as follows :

$$m_p(f) = \sqrt{2/\pi}\ \frac{1}{d_p} \int_0^{\pi} f(x)\, se_p(x, q)\, dx \qquad \ldots(1.1)$$

where $se_p(x, q)$ is the Mathieu function[3] of integral order p,

$$d_p = \left(\sum_{r=0}^{\infty} \left[B_r^{(p)}(q) \right]^2 \right)^{1/2},$$

$B_r^{(p)}$ is the value of q as indicated by (2.5) of Sharma and Gupta[5].

If $f(x)$ is sectionally continuous over a finite interval $I = (0, \pi)$ and satisfies the Dirichlet's conditions, then the inverse transform formula for the transform (1.1) is given by[5]

$$f(x) = \begin{cases} \sqrt{2/\pi} \sum_{n=0}^{\infty} \frac{m_{2n+1}[f(x)]\, se_{2n+1}(x, q)}{d_{2n+1}}, & p = 2n+1 \\ \sqrt{2/\pi} \sum_{n=0}^{\infty} \frac{m_{2n+2}[f(x)]\, se_{2n+2}(x, q)}{d_{2n+2}}, & p = 2n+2. \end{cases} \quad \ldots(1.2)$$

The present work is also an extension of some results of Sharma and Gupta[5] to a certain class of generalized functions. Finally, the transform developed in this paper has been applied to the problem of heat conduction between two parallel plates where the source of heat generation depends upon temperature linearly. The notations and terminology of the present work follow that of Sharma and Gupta[5] and Zemanian[6].

2. The Testing Function Space $M(I)$

Let $M(I)$ be a set of all complex-valued smooth functions $\phi(t)$ defined on $I = (0, \pi)$ such that

(i) $\gamma_k(\phi) = \gamma_0(R^k \phi) = [\int_0^{\pi} | R^k \phi(x) |^2 dx]^{1/2} < \infty$

$$k = 0, 1, 2 \ldots$$

(ii) For each pair of non-negative integers n and k

$$(R^k \phi, \psi_n) = (\phi, R^k \psi_n)$$

where R denote the differential operator

$$R = \frac{d^2}{dx^2} - 2q \cos(2x) \quad \ldots(2.1)$$

and $\psi_n(x) = \sqrt{2/\pi}\, se_n(x, q)$ and $(R^k \phi, \psi_n)$ is an inner product in $L_2(I)$ as indicated by Zemanian[6] (p. 249).

$M(I)$ is a linear space and since γ_0 is a norm, the collection of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ is separating as indicated by Zemanian[6] (p.8). Consequently, $\{\gamma_k\}_{k=0}^{\infty}$ is a

countable multinorm on $M(I)$. We assign to $M(I)$ the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$, and thus $M(I)$ is a countably multinormed space. Moreover, $M(I)$ is complete and therefore a Frechet space.

The dual of $M(I)$ is denoted by $M'(I)$. It can also be shown that $M'(I)$ is complete. Then $\{\psi_n\}_{n=0}^{\infty}$ forms a complete orthonormal system. Moreover, there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers called eigenvalues of R (corresponding to ψ_n) such that $|\lambda_n| \to \infty$ as $n \to \infty$ and

$$R\psi_n = \lambda_n \psi_n, \; n = 0, 1, 2, \ldots. \qquad \ldots(2.2)$$

Indeed, since the Mathieu functions $se_p(x, q)$ are solutions of the differential equation [Erdelyi[1], p. 111]

$$\frac{d^2u}{dx^2} + [h - 2q \cos 2x]\, u = 0 \qquad \ldots(2.3)$$

$$\frac{d^2\psi_n}{dx^2} + [h - 2q \cos 2x]\, \psi_n = 0$$

i. e.

$$R\psi_n = -h\psi_n \text{ by (2.1)}$$

$$= \lambda_n(q)\, \psi_n$$

where $\lambda_n(q)$ denotes the corresponding characteristic values of h [Erdelyi[1], p. 111]. Ince has shown that for $q \neq 0$,

$$|\lambda_n(q)| \to \infty \text{ as } n \to \infty \text{ [Erdelyi[1] p. 113].}$$

Every ψ_n is a member of $M(I)$. For, ψ_n is a complex-valued smooth function on I. Moreover,

$$[\gamma_k(\psi_n)]^2 = \int_0^{\pi} |R^k\psi_n|^2\, dx = |\lambda_n|^{2k} \int_0^{\pi} |\psi_n(x)|^2\, dx = |\lambda_n|^{2k}.$$

So that condition (i) is satisfied. Finally, if $n \neq m$, we have

$$(R^k\psi_n, \psi_m) = \lambda_n^k(\psi_n, \psi_m) = 0 = \lambda_m^k(\psi_n, \psi_m) = (\psi_n, R^k\psi_m)$$

whereas, for $n = m$,

$$(R^k\psi_n, \psi_n) = \left(\lambda_n^k \psi_n, \psi_n\right) = \left(\psi_n, \lambda_n^k \psi_n\right) = (\psi_n, R^k \psi_k)$$

since λ_n is real. Thus, condition (ii) is also satisfied.

We shall list a few properties of these spaces.

(a) $M(I)$ is a subspace of $L_2(I)$ when we identify each function in $M(I)$ with the corresponding equivalence class in $L_2(I)$. Convergence in $M(I)$ implies convergence in $L_2(I)$.

(b) $D(I)$ denotes the space of infinitely differentiable functions defined on I with compact support. It can be easily shown that $D(I)$ is a subspace of $M(I)$ and the topology of $D(I)$ is stronger than that induced on it by $M(I)$. Consequently, restriction of any $f \in M'(I)$ to $D(I)$ is in $D'(I)$. Moreover, convergence in $M'(I)$ implied convergence in $D'(I)$.

(c) For each $x \in I$, R is a continuous linear mapping of $M(I)$ into itself. In view of the self-adjoint nature of the operator R we define the generalized differential operator R on $M'(I)$ by

$$(Rf(x), \phi(x)) = (f(x), R\phi(x))$$

for $f \in M'(I)$, $\phi \in M(I)$.

(d) We imbed $L_2(I)$ (and therefore), $M(I)$ since $M(I)$ is a subspace of $L_2(I)$) into $M'(I)$ by defining the number that $f \in L_2(I)$ assigns to any $\phi \in M(I)$ as

$$(f, \phi) = \int_0^{\pi} f(x)\,\overline{\phi(x)}\,dx. \qquad \ldots(2.4)$$

Then f is linear and continuous on $M(I)$.

This imbedding of $L_2(I)$ into $M'(I)$ is one-to-one.

(e) If $f(x) = R^k g(x)$, for some $g \in L_2(I)$ and some k, then $f \in M'(I)$.

Instead of working with the number $\langle f, \phi \rangle$ that $f \in M'(I)$ assigns to $\phi \in M'(I)$, it is more convenient to work with the number that f assigns to the complex conjugate number of ϕ, we write

$$(f, \phi) = \langle f, \bar{\phi} \rangle. \qquad \ldots(2.5)$$

This is consistent with the inner product notation of $L_2(I)$.

The multiplication by a complex number a is given by

$$(af, \phi) = \langle af, \bar{\phi} \rangle = \langle f, a\bar{\phi} \rangle = a(f, \phi). \qquad \ldots(2.6)$$

The following Lemmas are useful in the subsequent analysis.

Lemma 2.1—If $\phi(x) \in M(I)$, then for $x \in I$

$$\phi(x) = \sum_{n=0}^{\infty} (\phi(x), \psi_n(x))\, \psi_n(x) \qquad \ldots(2.7)$$

where the series converges in $M(I)$.

PROOF : The proof is similar to that of Zemanian[6] (p. 255) and hence it is omitted.

Using (ii), (2.7) and the fact that the inner product is continuous with respect to each of its argument, for any two members ϕ and χ of $M(I)$, we obtain

$$(R\phi(x), \chi(x)) = (\phi(x), R(\chi(x))). \qquad \ldots(2.8)$$

Therefore the operator R is a self adjoint operator on $M(I)$. The next lemma is analogous to that of Lemma 9.3-3 (p.255) of Zemanian[6] and gives a characterization of orthonormal series that converges in $M(I)$.

Lemma 2.2—Let a_n denote complex numbers. Then $\sum_{n=0}^{\infty} a_n \psi_n(x)$ for $x \in I$, converges in $M(I)$ if and only if $\sum_{n=0}^{\infty} |\lambda_n|^{2k} |a_n|^2$ converges for every non-negative integer k.

3. ORTHONORMAL SERIES EXPANSION OF GENERALIZED FUCNTIONS

The following theorem provides an orthonormal series expansion of generalized functions belonging to $M'(I)$ which in turn yields an inversion formula for a certain generalized integral transformation.

Theorem 3.1—Let $f \in M'(I)$, then

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\, \psi_n \qquad \ldots(3.1)$$

where the series converges in $M'(I)$.

PROOF : By Lemma 2.1, for any $\phi \in M(I)$, we obtain

$$(f, \phi) = \left(f, \sum_{n=0}^{\infty} (\phi, \psi_n)\, \psi_n\right) = \sum_{n=0}^{\infty} (f, \psi_n)\, \overline{(\phi, \psi_n)}$$

(equation continued on p. 43)

$$= \sum_{n=0}^{\infty} (f, \psi_n)(\psi_n, \phi).$$

This proves that the series on the right hand side of (3.1) converges in $M'(I)$. Q. E. D.

The members of $M'(I)$ lead to the generalized Mathiew transformation m defined by

$$mf = F(n) = \left\langle f, \psi_n \right\rangle = (f, \psi_n), f \in M'(I), n = 0, 1, 2, 3... \quad ..(3.2)$$

Thus a linear and continuous mapping m maps $f \in M'(I)$ into a function $F(n)$ defined on the set of non-negative integers.

The inverse generalized Mathieu transformation m^{-1} is defined by the series (3.1) which we rewrite as

$$m^{-1} F(n) = \sum_{n=0}^{\infty} F(n) \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) = f. \quad ...(3.3)$$

Theorem 3.2 (*Uniquences*)—Let f and g be the members of $M'(I)$ and let their transforms $F(n)$ and $G(n)$ satisfy $F(n) = G(n)$ for all n, then $f = g$ in the sense of equality in $M'(I)$.

PROOF : We have

$$f - g = \Sigma (f - g, \psi_n) \psi_n = \Sigma [(f, \psi_n) - (g, \psi_n)] \psi_n = 0.$$

4. CHARACTERIZATION THEOREMS

The following theorem gives a characterization of the transform $F(n)$ of f in $M'(I)$ (i. e. $M'_n(f)$). Its proof being similar to that of Theorem 9.6-1, (p. 261) in Zemanian[6] and hence it is omitted.

Theorem 4.1—Let b_n denote complex numbers. Then for $x \in I$,

$$\Sigma b_n \psi_n (x) \quad ...(4.1)$$

converges in $M'(I)$ if and only if there exists a non-negative integer s such that the series

$$\sum_{\lambda_n \neq 0} |\lambda_n|^{-2s} |b_n|^2$$

converges. Furtheremore, if f denotes the sum of (4.1) then $b_n = (f, \psi_n)$.

The next theorem is analogous to that of Theorem 9.6-2 (p.262) in Zemanian[6] and gives a characterization of elements of $M'(I)$.

Theroem 4.2—A necessary and sufficient condition for f to be a member of $M'(I)$ is that there be some non-negative integer s and $a\ g \in L_2(I)$ such that

$$f(x) = R^s g(x) + \sum_{\lambda_n=0} C_n \psi_n(x),\ x \in I \qquad \ldots(4.2)$$

where C_n denotes the complex constants and $\sum_{\lambda_n=0}$ denotes the summation on those n for which $\lambda_n = 0$, there are only a finite number of such n.

5. Application

Our object is to solve the problem of heat conduction between two parallel insulated plates $x = 0$ and $x = \pi$ where the source of heat generation lies within it such that the distance π is much larger than the size of the source. The scale size of interest and the variation in the physical parameters are much larger than the gradient inside the source. We seek a conventional function $U(x, t)$ that depends on x and time t, and satisfies the differential equation

$$D_x^2\ U - 2q \cos(2x)\ U = K\ D_t U,\ U = U(x, t),\ 0 \leqslant x \leqslant \pi,\quad 0 < t < \infty \qquad \ldots(5.1)$$

with the following initial and boundary conditions.

(i) As $t \to 0+$, $U(x, t)$ converges to a certain generalized function $f(x)$.

(ii) $U(x, t) = 0$ for $x = 0$ and $x = \pi$.

We can solve this problem by using differential operator R defined in (2.1). The differential equation (5.1) now becomes

$$R(U) = K\ D_t(U),\ U = U(x, t),\ 0 \leqslant x \leqslant \pi,\ 0 < t < \infty \qquad \ldots(5.2)$$

whereas the initial and boundary conditions are taken to be

$$U(x, t) \to f(x) \text{ in } M'(I) \text{ as } t \to 0+ \qquad \ldots(5.3)$$

and

$$U(x, t) \to 0 \text{ at } x = 0 \text{ and } x = \pi. \qquad \ldots(5.4)$$

Let m denote the generalized Mathieu transformation on $M'(I)$ as defined by (3.2).

Then

$$m(RU) = -a_p(q)\, m\, U \qquad \ldots(5.5)$$

where $a_p(q)$ denote the corresponding characteristic values of h in differential equation (2.3) by virtue of Mathieu's equation McLachlan[3], [p.10].

Therefore, m transforms the differential equation (5.2) into

$$-a_p(q)\, mU = KD_t\, mU. \qquad \ldots(5.6)$$

Whose solution is

$$mU = A\, e^{-a_p(q)t/k}.$$

Using initial condition, we obtain

$$m[U(x, t)] = g(p)\, e^{-a_p(q)t/k}$$

where $g(p) = (f, \psi_p)$.

Now, by inverse transformation (3.3) we obtain the formal solution

$$U(x, t) = \sum_{p=0}^{\infty} g(p)\, e^{-a_p(q)t/k}\, \psi_p(x). \qquad \ldots(5.7)$$

$U(x, t)$ is conventional function that satisfies the differential equation (5.2). We now verify that (5.7) is a solution, i. e. (5.7) satisfies conditions (5.3) and (5.4). The condition (5.4) follows from the fact the $\psi_p(x)$ is a Mathieu function which is expanded in Fourier series as follows [Erdelyi[1], p. 115].

$$se_p(x, q) = \sum_{r=0}^{\infty} B_r^{(p)}(q) \sin(rx)$$

Finally, we note that, $g(p) = 0(|\lambda_p|^s)$, as $p \to \infty$ for some integer s, and by virtue of Lemma 2.2, the right-hand side of (5.7) converges in $M(I)$ and therefore in $L_2(I)$. Consequently we have

$$(U(x, t), \phi(x)) = \sum_{p=0}^{\infty} g(p)\, e^{-a_p(q)t/k}\, (\psi_p(x), \phi(x)). \qquad \cdots(5.8)$$

But $g(p)$ is of slow growth, while $(\psi_p(x), \phi(x))$ is of rapid descent as $p \to \infty$. So (5.8) converges uniformly on $0 < t < \infty$ and we may pass to the limit under the summation sign as $t \to 0+$ to get

$$(U(x, t), \phi(x)) \to \sum_{p=0}^{\infty} g(p)(\psi_p(x), \phi(x)) = (f, \phi).$$

Thus, initial condition (5.3) is also satisfied.

Remark

(i) Since, as $q \to 0$, $se_p(x) \to \sin(px)$, the Fourier finite sine transformation is a limiting case of the Mathieu transformation.

(2) The second Mathiew transform with an even Kernel $ce_p(x, q)$ may be developed for distribution as well. In this case, the Fourier cosine transformation will be a limiting case.

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