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CONTINUED FRACTIONS AND CERTAIN FUNCTIONAL EQUATIONS

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Application of continued fractions is made to a functional equation in a Hilbert Space endowed with the Dirichlet norm. The Neumann series solution for the Fredholm-Poincaré integral equation, whose kernel is fully symmetrizable by a positive definite operator, yields a Taylor's series with Schwarz's constants as coefficients.

Interesting results are established between coefficients associated with Lanczos minimized iteration scheme and those of the *J*-fraction and *S*-fraction expansions. Equivalence transformations help discuss the notions of bounds, convergence and truncation errors. As an Alternate to Fredholm theory, the meromorphic character of the solution is established with the help of the properties of continued fractions.

1. INTRODUCTION

Lyusternik[3] applied continued fractions to the eigen-value problem of a symmetric matrix and Stesin[8,9] investigated the case of a symmetric kernel. The latter showed that the successive denominators of the *S*-fraction expansion[7] corresponding to the Taylor series, approximate the Fredholm denominator of the Kernel, in the sense that the zeros of the successive denominators of the *S*-fraction tend towards the negative reciprocals of the eigen-values. He also arrived at certain error bounds with the help of Stieltjes' integrals.

Herein, the above analysis is extended to the kernel $K(p, q)$ of the integral equation associated with the Robin-Poincaré problem which belongs to symmetrizable class of kernels[4]. In an earlier paper[5] we discussed this equation and outlined the properties of K. We recall that K is fully symmetrizable by a positive definite kernel $G(p, q)$ which is itself symmetric in the spatial co-ordinates p and q.

2. FUNCTIONAL EQUATION

The Robin-Poincaré problem, the induced potential problem or more commonly the Neumann boundary value problem in three dimensions leads to the solution of the following Fredholm-Poincaré integral equation

$$\mu = f + \lambda K\mu \qquad \ldots(2.1)$$

where $f(p)$ is the value of the normal derivative of the potential on the closed smooth surface S of class B (ref. 4) with respect to which the problem is formulated and λ is a parameter.

The solution $\mu(p)$ of (2.1) corresponds to the density of the single layer, the potential due to which gives the harmonic function sought.

If γ_{pq} is the distance between the points p and q on S and n_p is the outward normal at p, then the kernel K of (2.1) is

$$K(p, q) = \frac{\partial}{\partial n_p} G(p, q) \qquad \ldots(2.2)$$

where,

$$G(p, q) = \frac{1}{2\pi} \cdot \frac{1}{\gamma pq} . \qquad \ldots(2.3)$$

We have discussed in detail the properties of (2.2) in an earlier work[5].

Let $f(p)$ be a real function continuous for $p \in S$, and H the corresponding Hilbert space, then

$$K : H \to H \text{ and } G : H \to H$$

are linear transformations in H. If $f, g \in H$, then in usual notation

$$(f, g) = \int_S f(p).\ g(p)\, dS_p.$$

We define a scalar product in H such that

$$[f, g] = (Gf, g). \qquad \ldots(2.4)$$

G is symmetric and positive definite, i. e.

$$(Gf, g) = (f, Gg);\ (Gf, f) > 0.$$

GK is a symmetric transformation and K is symmetric in the new scalar i. e.,

$$[Kf, g] = [f, Kg]. \qquad \ldots(2.5)$$

The Dirichlet norm of $f \in H$ is defined as

$$\|f\| = [f, f]^{1/2} = (J + J')^{1/2} > 0, \qquad \ldots(2.6)$$

where J and J' are Dirichlet integrals for R and R', the regions interior and exterior to S, respectively and are essentially positive quantities. Also,

$$[Kf, f] = J - J' \qquad \ldots(2.7)$$

whence,

$$[Kf, f] < [f, f]. \qquad \ldots(2.8)$$

We recall[5] that K satisfies the Fredholm alternative for all values of λ and that it has a discrete spectrum with no finite point of accumulation. The set $\left\{\phi_i, \lambda_i^{-1}\right\}$ of eigen-functions and eigen -values of K is denumerably infinite where $\lambda_i^{-1} \to 0$ and $\{\phi_i\}$ is the maximal set of linearly independent orthogonal eigen-functions, ϕ_i corresponding to positive eigen-value λ_i^{-1} and

$$\lambda_1^{-1} > \lambda_2^{-1} \ldots > \lambda_n^{-1} \ldots \qquad \ldots(2.9)$$

3. Schwarz's Constants and Lanczos Coefficients

A Taylor's series solution of (2.1) may be formally written as

$$\mu = (1 - \lambda K)^{-1} f \qquad \ldots(3.1)$$

i. e.

$$\mu = f + \lambda Kf + \lambda^2 K^2 f + \lambda^3 K^3 \ldots \qquad \ldots(3.2)$$

in the event $\|\lambda K\| < 1$.

or

$$\|K\| < \frac{1}{\lambda}. \qquad \ldots(3.3)$$

We may take the scalar product of (3.2) with f and denoting the Schwarz's constants by

$$c_n = [K^n f, f], n = 0, 1, \ldots. \qquad \ldots(3.4)$$

write the resulting polynomial in λ as

$$P(\lambda) = [\mu, f] = c_0 + \lambda c_1 + \lambda^2 c_2 + \lambda^3 c_3. \qquad \ldots(3.5)$$

Furthermore, let

$$f_0, f_1, f_2, \ldots, f_0 = f \qquad \ldots(3.6)$$

be the orthogonal functions obtained according to the minimized iteration scheme[2,4]. Then the Lanczos coefficients α's and β's are given by

$$\alpha_n = \frac{[Kf_n, f_n]}{[f_n, f_n]}$$

$$\beta_n = \frac{[Kf_{n+1}, f_n]}{[f_n, f_n]} = \frac{[f_{n+1}, Kf_n]}{[f_n, f_n]} = \frac{[f_{n+1}, f_{n+1}]}{[f_n, f_n]} > 0. \qquad \ldots(3.7)$$

Theorem 1—The polynomial $P(\lambda)$ is such that Δ_n defined by (3.8) below are strictly positive for all n.

We note that the β's are strictly positive. As a consequence of this the principle minors of the determinant Δ_n having elements as Schwarz's constants c_n,

$$\Delta_n = \begin{vmatrix} c_0 & c_1 \ldots & c_n \\ c_1 & c_2 \ldots & \\ . & . & \\ . & . & \\ c_n & c_{n+1} & c_{2n} \end{vmatrix} \qquad \ldots(3.8)$$

are strictly positive.

This can easily be seen in view of the fact[6] that :

$$\Delta_0 = c_0 = [f_0, f_0] > 0$$

$$\Delta_1 = \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} = c_0^2 \quad \beta_0 > 0.$$

and

$$\Delta_2 = \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix} = c_0^3 \quad \beta_0^3 \quad \beta_1 > 0.$$

etc. Hence, $\Delta_n > 0$ for all n.

Theorem 2—For the set of orthogonal functions $\{f_i\}$ generated by Lanczos minimized interaction scheme, Bessels' inequality holds strictly, i. e.,

$$[Kf_1, f_1]^2 < [Kf_1. \quad Kf_1], [f_1, f_1] \qquad \ldots(3.9)$$

etc.

This can easily be seen with the help of the analogue of the scalar properties[5]. We recall that

$$\frac{[Kf_n,\ Kf_n]}{[f_n,\ f_n]} = \alpha_n^2 + \beta_n + \beta_{n-1} \qquad \ldots(3.10)$$

where each quantity on the right is positive. Unless $\beta_n = \beta_{n-1} = 0$, we have

$$\alpha_n^2 < \frac{[Kf_n,\ Kf_n]}{[f_n,\ f_n]}.$$

Substituting the value of α_n from (3.7) we have

$$[Kf_n,\ f_n]^2 < [Kf_n,\ Kf_n].\ [f_n, f_n]. \qquad \ldots(3.11)$$

Similarly, we have

$$\beta_n < \frac{[Kf_n,\ Kf_n]}{[f_n, f_n]} \qquad \ldots(3.12)$$

or, by (3.7)

$$[f_{n+1}, f_{n+1}] < [Kf_n,\ Kf_n]$$

and also

$$[f_n, f_n]^2 < [Kf_n,\ Kf_n]\,[f_{n-1}, f_{n-1}]. \qquad \ldots(3.13)$$

Since K is fully symmetrizable[5] the condition

$Kf_n = 0$ may be interpreted to mean that

$$\alpha_n = \beta_n = \beta_{n-1} = 0$$

and vice-versa. When K is finite dimensional matrix A, this would be automatically satisfied.

4. *S*-Fraction Expansion

Suppose that $P(\lambda)$ as given by (3.5) has the following S-fraction expansion

$$\cfrac{b_0}{1} - \cfrac{b_1\lambda}{1} - \cfrac{b_2\lambda}{1} - \cfrac{b_3\lambda}{1} - \cdots. \qquad \ldots(4.1)$$

The coefficients of partial numerators above then have the following values in terms of Lanczos' coefficients,

$$b_0 = c_0,\ b_1 = \alpha_0,\ b_2 = \beta_0/\alpha_0. \qquad \ldots(4.2)$$

These results may be summarised as in Nayar[6].

Theorem 3—The coefficients of *S*-fraction expansion of $P(\lambda)$ are given by the following recurrence relations

$$b_{2n-1} = D_{n-1}/D_{n-2} \qquad \ldots(4.3)$$

$$b_{2n} = \beta_{n-1}\,(D_{n-2}/D_{n-1})$$

where, D_n is the determinant

$$D_n = \begin{vmatrix} \alpha_0 & 1 & 0 & 0\,. & 0 \\ \beta_0 & \alpha_1 & 1 & 0\,. & \\ 0 & \beta_1 & \alpha_2 & 1 & \\ 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & \beta_{n-1} & \alpha_n \end{vmatrix} \quad \begin{matrix} D_0 = \alpha_0, \\ D_{-1} = 1, \end{matrix} \qquad \ldots(4.4)$$

corresponding to the tridiagonal matrix representation of order $(n+1)$ of the kernel K.

Corollary 3.1—The coefficients $\{b_l\}$ of the *S*-fraction are connected by α's and β's by the recurrence relations

$$b_{2n+1} + b_{2n} = \alpha_n$$

and $\qquad \ldots(4.5)$

$$b_{2n+1} - b_{2n} = \beta_{n-1},\ \beta_l > 0.$$

To facilitate further computations, the *S*-fraction (4.1) is transformed into a more convenient form through a series of equivalence transformations[10]. (i) Changing λ to $-\frac{1}{\lambda}$ and multiplying by $\frac{1}{\lambda}$ we get,

$$\sum \frac{(-1)^n C_n}{\lambda^{n+1}} = \frac{b_0\,|}{|\,\lambda} + \frac{b_1\,|}{|\,1} + \frac{b_2\,|}{|\,\lambda} + \frac{b_3}{1}. \qquad \ldots(4.6)$$

(ii) Putting $\xi^2 = \lambda$, (ξ has positive real part), the continued fraction becomes

$$\frac{b_0\xi^{-1}\,|}{|\,\xi} + \frac{b_1\,|}{|\,\xi} + \frac{b_2\,|}{|\,\xi}. \qquad \ldots(4.7)$$

(iii) Reduction of partial numerators to unity gives

$$\frac{\xi^{-1}\,|}{|\,b_1'\,\xi} + \frac{1\,|}{|\,b_2'\,\xi} + \frac{1\,|}{|\,b_3'\,\xi} \qquad \ldots(4.8)$$

where

$$b_1' = \frac{1}{b_0}, \quad b_2' = \frac{b_0}{b_1}, \quad b_3' = \frac{b_1}{b_0 b_2}. \qquad \ldots(4.9)$$

Since (4.8) will be used quite a bit, we may state the following :

Theorem 4—The coefficients $\{b_i'\}$ of the continued fraction with partial numerators unity, are given by the relations

$$b_{2n-1}' = \frac{b_1 b_3 \ldots\ldots b_{2n-3}}{b_0 b_2 \ldots\ldots b_{2n-2}} \qquad \ldots(4.10)$$

and

$$b_{2n}' = \frac{b_0 b_2 \ldots\ldots b_{2n-2}}{b_1 b_3 \ldots\ldots b_{2n-1}}$$

such that $b_i' > 0$.

Observe that $b_{2n-1}' > 0$, since by (4.5), β's are positive and

$$b_{2n-1}' = \frac{b_1^2 \quad b_3^2 \ldots\ldots b_{2n-3}^2}{b_0 b_1 b_2 b_3 b_4 \ldots b_{2n-3} b_{2n-2}} = \frac{b_1^2 \quad b_3^2 \ldots b_{2n-3}^2}{C_0 \beta_0 \beta_1 \ldots \beta_{n-2}} \qquad \ldots(4.11)$$

also,

$$b_{2n}' = \frac{c_0 \beta_0 \beta_1 \ldots \beta_{n-2}}{b_1^2 \quad b_3^2 \quad \ldots \quad b_{2n-3}^2} \cdot \frac{1}{b_{2n-1}}. \qquad \ldots(4.12)$$

Thus the sign of b_{2n}' is governed by that of b_{2n-1} which by (4.3) is the ratio D_{n-1}/D_{n-2}. Since D_n is associated with a positive definite matrix, we see that the principal minors $D_i > 0$ and consequently $b_{2n}' > 0$. However, we may try to argue further on the basis of (4.11) which is conclusive atleast.

5. Recurrence Relations : Ceratain Inequalities

Let us deal with the S-fraction in ξ as given by (4.8) and denote its nth approximant by P_n/Q_n. The following recurrence relations are easily obtainable :

$$P_n = b'_n \xi P_{n-1} + P_{n-2}$$

$$Q_n = b'_n \xi Q_{n-1} + Q_{n-2} \qquad \ldots(5.1)$$

with

$$P_0 = 0, Q_0 = 1, P_1 = \xi^{-1}, Q_1 = b'_1 \xi \ldots$$

The relative difference of the two consecutive approximants may be noted as :

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{-(P_{n-1} Q_{n-2} - P_{n-2} Q_{n-1})}{Q_n Q_{n-1}} . \qquad \ldots(5.2)$$

The numerator on the right simplifies to $(-1)^n P_1$,
thus we have

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}}{Q_n Q_{n-1}} \qquad \ldots(5.3)$$

which is a telescopic alternating series and can be summed up to give

$$\frac{P_n}{Q_n} = \xi^{-1} \sum_{n=1}^{n} \frac{(-1)^{n-1}}{Q_{n-1} Q_n} . \qquad \ldots(5.4)$$

We now seek a lower bound for the denominator of the general term on the right above. Let,

$$X_n = Q_n Q_{n-1}. \qquad \ldots(5.5)$$

Multiplying the second recurrence relation in (5.1) by X_{n-1},

we have $X_n - X_{n-1} = b'_n \xi Q^2_{n-1}$. $\qquad \ldots(5.6)$

As before, by adding (5.6) we have

$$X_n = \xi \sum_{n=1}^{n} b'_n Q^2_{n-1}. \qquad \ldots(5.7)$$

Consequently,

$$X_n - X_{n-2} = \xi \left(b'_{n-1} Q^2_{n-2} + b'_n Q^2_{n-1} \right). \qquad \ldots(5.8)$$

Using the inequality between arithmetic and geometric mean, the above may be written as

$$X_n \geqslant X_{n-2} + 2\xi \sqrt{b'_{n-1} \quad b'_n \quad X_{n-1}}. \qquad \ldots(5.9)$$

Theorem 5—If $Q_{2n}(\xi)$ denote the successive denominators of the S-fraction (4.8) and $X_{2n} = Q_{2n}\ Q_{2n-1}$, then $\{X_{2n}\}$ is bounded from below by (5.11).

Employing (4.3), (4.11), (4.12) and (5.6) by changing the index n to $2n-1$ we get

$$X_{2n-1} - X_{2n-2} = \xi \left\{ b'_{2n-1} \quad Q^2_{2n-2} \right\} > 0. \qquad \ldots(5.10)$$

Likewise from (5.9) changing index n to $2n$ we get the following inequality,

$$X_{2n} \geqslant \left[1 + 2\xi \left(\frac{D_{n-2}}{D_{n-1}} \right)^{1/2} \right] X_{2n-2}. \qquad \ldots(5.11)$$

Corollary 5.1—$\{X_{2n}\}$ forms an increasing sequence.

When n is odd (5.10) gives

$$X_{2n+1} > X_{2n}$$

or

$$\frac{1}{X_{2n+1}} < \frac{1}{X_{2n}}. \qquad \ldots(5.12)$$

6. Bounds for Even and Odd Approximants

Theorem 6—The even and odd approximants of (4.8) are bounded from above by $1/\xi\, X_1$.

From (4.5) and (5.5) we have

$$\frac{P_n}{Q_n} = \xi^{-1} \left(\frac{1}{X_1} - \frac{1}{X_2} - \frac{1}{X_3} \cdots \frac{(-1)^{n-1}}{X_n} \right) \qquad \ldots(6.1)$$

and likewise,

$$\frac{P_{2n+1}}{Q_{2n+1}} = \xi^{-1} \left[\frac{1}{X_1} - \left(\frac{1}{X_2} - \frac{1}{X_3} \right) - \left(\frac{1}{X_4} - \frac{1}{X_5} \right) - \left(\frac{1}{X_{2n}} - \frac{1}{X_{2n+1}} \right) \right]. \qquad \ldots(6.2)$$

For n odd, by virtue of (5.12), each difference in (6.2) is positive and hence the odd approximants are bounded from above.

$$\left|\frac{P_{2n+1}}{Q_{2n+1}}\right| < \frac{\xi^{-1}}{X_1} . \qquad \ldots(6.3)$$

For n even,

$$\left|\frac{P_{2n}}{Q_{2n}}\right| = \xi^{-1}\left[\frac{1}{X_1} - \left(\frac{1}{X_2} - \frac{1}{X_3}\right) \cdots \left(\frac{1}{X_{2n-2}} - \frac{1}{X_{2n-1}}\right) - \frac{1}{X_{2n}}\right]. \qquad \ldots(6.4)$$

As before, each difference in the bracket is positive and by virtue of (5.11), $X_{2n} > 0$, we have

$$\left|\frac{P_{2n}}{Q_{2n}}\right| < \frac{\xi^{-1}}{X_1} . \qquad \ldots(6.5)$$

Corollary 6.1—The relative difference of two successive approximants of (4.8) is bounded.

From (5.3) and (5.5) by changing the index, we get

$$\left|\frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}}\right| = \frac{\xi^{-1}}{X_{2n}} . \qquad \ldots(6.6)$$

Theorem 7—The bound of the relative difference of two successive approximants of (4.8) can be expressed in terms of Schwarz's constants and parameter λ of the functional equation (2.1).

By repeated application of (5.11) using Minkowski inequality and supplying an extra nth factor, we get

$$X_{2n} \geqslant \left\{1 + 2\xi\left(\frac{1}{D_{n-1}}\right)^{1/2n}\right\} X_2 \Big/ \left\{1 + 2\xi\left(\frac{D_{-1}}{D_0}\right)^{1/2}\right\}. \qquad \ldots(6.7)$$

Since $D_{-1} = 1$ and $D_0 = \alpha_0 = b_1$, and (5.1) for $n = 2$, we get by using the values of Q_2, Q_1 and b_1' b_2' ,

$$X_2 = (\alpha_0 + \xi^2)\,\xi/\alpha_0 c_0. \qquad \ldots(6.8)$$

Whence from (6.6), (6.7) and (6.8) is follows that

$$\left|\frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}}\right| < \frac{\alpha_0 c_0\,(1 + 2\,\xi\,\alpha_0^{-1/2})}{\xi^2\,(\alpha_0 + \xi^2)}[1 + 2\xi\,(D_{n-1})^{-1/2n}]^{-n} \qquad \ldots(6.9)$$

We recall an earlier result[6].

$$D_{n-1} = B_n/A_n \qquad \ldots(6.10)$$

where,

$$A_n = \begin{vmatrix} c_0 & c_1 & - & - & c_{n-1} \\ c_1 & c_2 & - & - & c_n \\ c_{n-1} & c_n & - & & c_{2n-2} \end{vmatrix}; \; B_n = \begin{vmatrix} c_1 & - & - & c_n \\ , & & & \\ c_n & - & - & c_{2n-1} \end{vmatrix}. \qquad \ldots(6.11)$$

Now $\lambda = \xi^2$, consequently (6.9) gives the following

$$\left| \frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}} \right| \leqslant \frac{c_0 c_1 \left(1 + 2 \sqrt{\frac{c_0 \lambda}{c_1}} \right)}{\lambda (c_1 + c_0 \lambda)} \times \left[1 + 2 \sqrt{\lambda} \left(\frac{A_n}{B_n} \right)^{1/2n} \right]^{-n} \qquad \ldots(6.12)$$

In the next section an estimate is made of the order of magnitude of the term on the right of (6.12).

7. Order of Magnitude

In this section we seek the order of magnitude of certain quantities of interest and apply Hadamard's notion of limit superior[1] to the series of positive terms

$$c_1, \sqrt{c_2}, \ldots \sqrt[n]{c_n} \ldots. \qquad \ldots(7.1)$$

The c_i are coefficients of the polynomial $P(\lambda)$ given by (3.5). If it has a positive radius of convergence λ_0, then for sufficiently large n,

$$\frac{1 - \epsilon}{\lambda_0} < \sqrt[n]{c_n} < \frac{1 + \epsilon}{\lambda_0} \qquad \ldots(7.2)$$

where ϵ is an arbitrary small positive quantity.

It is known that by the technique of analytic continuation, the radius of convergence can be increased by multiplication with a suitable polynomial, say of degree m which removes the singularities and yields a function which is regular in a circle of radius λ', where $\lambda_0 < \lambda'$.

The coefficients of this new series would then satisfy inequalities of the type (7.2) λ_0, being replaced by λ'. Thus the estimates of the determinants of order $m + 1$ denoted by

$$D_n^{(m)} = \begin{vmatrix} c_n & c_{n+1} & \cdot & \cdot & c_{n+m} \\ \cdot & & & & \\ \cdot & & & & \\ c_{n+m} & \cdot & \cdot & \cdot & c_{n+2m} \end{vmatrix} \quad \ldots(7.3)$$

can be made

Theorem 8—If λ_0^{-1} is the first eigen value-with repetition m and λ'^{-1} the next, then the order of magnitude of (7.3) for different m and n can be estimated in terms of λ_0 and λ' as

$$\left| D_n^{(m)} \right| < \left(\frac{1+\epsilon}{\lambda_0^m \lambda'} \right)^n \quad \ldots(7.4)$$

and

$$\left| D_{n-1}^{(m)} \right| > \left(\frac{1-\epsilon}{\lambda_0^m \lambda'} \right)^{n-1}. \quad \ldots(7.5)$$

The estimates of determinants of lower order than m, say of order $m-1$, $m-2$, etc., can similarly be made e. g.

$$\left| D_n^{(m-1)} \right| > \left(\frac{1-\epsilon}{\lambda_0^m} \right)^n \quad \ldots(7.6)$$

and

$$\left| D_n^{(m-2)} \right| < \left(\frac{1+\epsilon}{\lambda_0^{m-1}} \right)^n. \quad \ldots(7.7)$$

From (7.4) and (7.5), we have

$$\frac{D_n^{(m)}}{D_{n-1}^{(m)}} < \left(\frac{1+\epsilon}{1-\epsilon} \right)^n \cdot \frac{1-\epsilon}{\lambda_0^m \lambda'}$$

$$< \left(\frac{1+\epsilon}{1-\epsilon} \right)^n \cdot \frac{1-\epsilon}{\lambda_0^{m+1}}. \quad \ldots(7.8)$$

Since

$$\frac{1}{\lambda'} < \frac{1}{\lambda_0}.$$

Theorem 9—The relative difference of the two successive approximants in (6.9) tends to zero as n tends to infinity yielding a convergent sequence of approximants.

From (6.10), (6.11) and (7.3) we have

$$D_m = \frac{B_{m+1}}{A_{m+1}} = \frac{D_1^{(m)}}{D_0^{(m)}} < \left(\frac{1+\epsilon}{\lambda_0^{m+1}} \right) \qquad \ldots(7.9)$$

in the last inequality we have used (7.8) for $n = 1$. Similarly, using (7.6)

$$D_m^{-1} = \frac{D_0^{(m)}}{D_1^{(m)}} < \frac{\lambda_0^{m+1}}{1-\epsilon} \qquad \ldots(7.10)$$

whence by changing m to $n-1$, we have

$$\left(D_{n-1}^{-1} \right)^{1/2n} < \left[\lambda_0^n / (1-\epsilon) \right]^{1/2n} < \frac{\sqrt{\lambda_0}}{(1-\epsilon)^{1/2n}}. \qquad \ldots(7.11)$$

Employing this in (6.12) we get

$$\left| \frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}} \right| < \frac{c_0 c_1 \left(1 + 2\sqrt{\frac{\lambda c_0}{c_1}} \right)}{\lambda (c_1 + c_0 \lambda)} \left[1 + \frac{2\sqrt{\lambda \lambda_0}}{(1-\epsilon)^{1/2n}} \right]^{-n} \qquad \ldots(7.12)$$

The right-hand side above tends to zero as $n \to \infty$. Thus the sequence of the approximants is convergent. Hence continued fraction (4.8) converges.

8. Estimate of Lanczos Coefficients

In this section we give an order of magnitude of α's and β's of section 3.

Theorem 10—The coefficients β_r of Lanczos are strictly positive having modulus less than unity.

Now

$$\beta_{m-1} = \frac{A_{m+1} \cdot A_{m-1}}{A_m^2} = \frac{D_0^{(m)} \cdot D_0^{(m-2)}}{\left[D_0^{(m-1)} \right]^2}. \qquad \ldots(8.1)$$

Where $D_n^{(n)}$, defined by (7.3), is a determinant of order $m + 1$, starting with first term c_n, the coefficient of the Taylor's series, c_l are the Schwarz's constants defined by (3.4).

If λ_0^{-1} is the eigen-value with repetition m, and λ'^{-1} is the next higher eigen-value then employing the bounds of the various determinants using Hadamards results on limit superior as in (7.4) and (7.5) above we have

$$\frac{D_{n-1}^{(m)} \cdot D_{n-1}^{(m-2)}}{\left[D_{n-1}^{(m-1)} \right]^2} < \left[\sqrt{\frac{\lambda_0}{\lambda'}} \left(\frac{1+\epsilon}{1-\epsilon} \right) \right]^{2(n-1)} = \theta^{2(n-1)} \qquad \ldots(8.2)$$

where

$$\theta = \sqrt{\frac{\lambda_0}{\lambda'}} \frac{1+\epsilon}{1-\epsilon}. \qquad \ldots(8.3)$$

Putting $n = 1$ in (8.2) we have from (8.1)

$$\rho_{m-1} < 1. \qquad \ldots(8.4)$$

We may estimate the order of the magnitude for coefficients of the S-fraction expansion as well.

Theorem 11—The coefficients $\{b_l\}$ of S-fraction (4.6) are strictly positive and bounded from above. The bounds may be expressed in terms of λ_0, λ' and non-dimensional quantity θ.

$$b_{2m} = \frac{A_{m+1} \cdot B_{m-1}}{A_m B_m} = \frac{D_0^{(m)} D_1^{(m-2)}}{D_0^{(m-1)} D_1^{(m-1)}}. \qquad \ldots(8.5)$$

Now, using (7.4) and (7.5) for different indices, we have

$$\frac{D_{n-1}^{(m)} \cdot D_n^{(m-2)}}{D_{n-1}^{(m-1)} \cdot D_n^{(m-1)}} < \left(\frac{1-\epsilon}{1+\epsilon} \right) \lambda' \theta^{2n}. \qquad \ldots(8.6)$$

Whence we have from (8.5) and (8.6) for $n = 1$

$$b_{2m} < \left(\frac{1-\epsilon}{1+\epsilon} \right) \lambda' \theta^2 = \left(\frac{1+\epsilon}{1-\epsilon} \right) \lambda_0 = \sqrt{\lambda_0 \lambda'}\, \theta \qquad \ldots(8.7)$$

where we have used (8.3) above.

Similarly, we may estimate the odd coefficients.

$$b_{2m+1} = \frac{A_m . B_{m+1}}{A_{m+1} . B_m} = \frac{D_0^{(m-1)} D_1^{(m)}}{D_0^{(m)} D_1^{(m-1)}} \qquad ...(8.8)$$

by using the values for different indices m and n in (7.4) and (7.5). We have

$$\frac{D_{n-1}^{(m-1)} D_n^{(m)}}{D_{n-1}^{(m)} . D_n^{(m-1)}} < \left(\frac{1+\epsilon}{1-\epsilon}\right)^{2n-1} . \frac{1}{\lambda'} \qquad ...(8.9)$$

which yields as before for $n = 1$

$$b_{2m+1} < \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{\lambda'} = \frac{1}{\sqrt{\lambda_0 \lambda'}} \theta. \qquad ...(8.10)$$

Corollary 11.1—The coefficients $\{b_t\}$ are such that the even one's have the dimensions of λ and the odd ones of λ^{-1}.

Thus from (8.7) and (8.10), we have by multiplication

$$b_{2m} . b_{2m+1} < \theta^2 \qquad ...(8.11)$$

also by addition,

$$b_{2m} + b_{2m+1} < \left(\sqrt{\lambda_0 \lambda'} + \frac{1}{\sqrt{\lambda_0 \lambda'}}\right) \theta. \qquad ...(8.12)$$

Theorem 12—The coefficients α_t, β_t of Lanczos are such that $\alpha_m < 2\theta$ and $\beta_m < \theta^2$, where m indicates the multiplicity of the eigen value λ_0^{-1} .

By changing the index in (4.5) we get

$$b_{2m} + b_{2m+1} = \alpha_m. \qquad ...(8.13)$$

Substituting the value of b_{2m+1} from the above inequality (8.11), we get

$$\alpha_m < b_{2m} + \frac{\theta^2}{b_{2m}}$$

i. e.

$$b_{2m}^2 - \alpha_m b_{2m} + \theta^2 > 0 \qquad ...(8.14)$$

which may be written as

$$(b_{2m} - \theta)^2 + b_{2m}(2\theta - \alpha_m) > 0$$

since b_{2m} are positive, the above inequality will hold if

$$\alpha_{2m} < 2\theta. \qquad \ldots(8.15)$$

Furthermore, we know that

$$b_{2m+1c}\ b_{2m+2} = \beta_m. \qquad \ldots(8.16)$$

Using (8.7) for index $m + 1$ and and (8.10), we get

$$\beta_m < \theta^2. \qquad \ldots(8.17)$$

9. Some Equivalence Transformations

We recall[6] that the coefficients of the continued fraction in which the partial numerators are unity are given by

$$b'_{2n-1} = \frac{A_{n-1} \cdot B_n}{A_n \cdot B_{n-1}}$$

and

$$b'_{2n} = \frac{A_{n+1} \cdot B_{n-1}}{A_n \cdot B_n}$$

where A_n and B_n are defined in (6.11). These are equivalent to

$$b'_{2n-1} = \frac{b_1 b_3 - - - b_{2n-3}}{b_0 b_2 - - - b_{2n-2}} \qquad \ldots(9.1)$$

and

$$b'_{2n} = \frac{b_0 b_2 - - b_{2n-2}}{b_1 b_3 - - - b_{2n-1}}$$

respectively.

We may state the following :

Theorem 13—The coefficients $\left\{ b'_i \right\}$ of the continued fraction (4.8) are related to $\{b_i\}$ of (4.7) through equations (9.2) and (9.3).

Evidently from (4.5) and (4.11) or (9.1) above,

we have

$$b'_{2n-1} \cdot b'_{2n} = \frac{1}{b_{2n-1}} = \frac{b_{2n}}{\beta_{n-1}}. \qquad \ldots(9.2)$$

Also

$$b'_{2n-1} \cdot b'_{2n} = \frac{1}{b_{2n-1}} = \left(\frac{D_{n-2}}{D_{n-1}} \right) \qquad \ldots(9.3)$$

where, D_n is defined in (4.4). The recurrence relation for it is the following,

$$D_n = \alpha_n D_{n-1} - \beta_{n-1} D_{n-2}. \qquad \ldots(9.4)$$

Corollary 13.1—The coefficients $\{ b'_i \}$ are such that the product $b'_{2n-1} \cdot b'_{2n}$ is bounded from above.

By virtue of the positive definite nature of the matrix D_n, the necessary and sufficient conditions yield that the determinants D_n are all positve. Thus from (9.4) we get

$$\frac{D_{n-2}}{D_{n-1}} < \frac{\alpha_n}{\beta_{n-1}} \qquad \ldots(9.5)$$

whence from (9.3) and (8.15)

$$b'_{2n-1} \cdot b'_{2n} = \frac{1}{b_{2n-1}} < \frac{\alpha_n}{\beta_{n-1}} < \frac{2\theta}{\beta_{n-1}} . \qquad \ldots(9.6)$$

Corollary 13.2—The coefficients $\{b_i\}$ are such that they are bounded from above and below.

From the foregoing, by changing the index n to $m + 1$ we get

$$\frac{\beta_m}{\alpha_{m+1}} < b_{2m+1} < \frac{1}{\sqrt{\lambda_0 \lambda'}} \theta \qquad \ldots(9.7)$$

by using (9.6) and (8.10). By virtue of (8.3) the right inequality gives

$$b_{2m+1} < \frac{1}{\lambda'} \left(\frac{1 + \epsilon}{1 - \epsilon} \right) \qquad \ldots(9.8)$$

which is evident from (8.10) and (8.3).

Theorem 14—$P(\lambda)$ satisfies the necessary and sufficient condition for being a meromorphic function in the whole plane.

By repeated application of the inequality

$$D_n < \alpha_n D_{n-1} \qquad \ldots(9.9)$$

which can easily be seen by neglecting the negative term on the right of (9.4), we have

$$D_n < \alpha_0 \alpha_1 \ldots \alpha_n \qquad \ldots(9.10)$$

and since each α_n is less than 1,

$$D_n < 1. \qquad \ldots(9.11)$$

Also from (4.3) and (9.5), we get

$$b_{2n} = \beta_{n-1} \times \frac{D_{n-2}}{D_{n-1}} < \beta_{n-1} \times \frac{\alpha_n}{\beta_{n-1}} = \alpha_n. \qquad \ldots(9.12)$$

Since $\alpha_n \to 0$. For a finite dimensional matrix, $\alpha_n = 0$ automatically.

$$b_{2n} = 0 \qquad \ldots(9.13)$$

i. e. the continued fraction terminates.

Using properties of continued fractions detailed in the earlier paper[6], we conclude that $P(\lambda)$ is meromorphic in the whole plane. The condition $b_{2n} \to 0$ as $n \to \infty$ being both necessary and sufficient.

This is an important property of the Kernel K of the integral equation (2.1). Since the computation of b_l can be carried out numerically the method outlined above gives an alternate to Fredholm theory. If the first eigenvalue is m-tuple, then corresponding to (8.10) of Theorem 11 an identical result to (9.8) can easily be deduced for the coefficients b_l, i. e.

$$b_{2m-1} < \frac{1}{\lambda_0} \left(\frac{1+\epsilon}{1-\epsilon} \right) \qquad \ldots(9.14)$$

where λ_0^{-1} is the first eigenvalue.

Corollary 14.1—The quantities α_l, β_l and D_l are all positive with modulli less than unity. Specifically.

$$D_n < \alpha_0 \alpha_1 \ldots \alpha_n.$$

and D_{n-1} are bounded.

By repeated application of inequality (9.5) we get

$$\frac{\beta_0\,\beta_1\,.\,.\,\beta_{n-1}}{\alpha_1\,\alpha_2\,.\,.\,\alpha_n} < D_{n-1} < 1 \qquad \ldots(9.15)$$

where we have used (9.11) above. Whence

$$\beta_0\,\beta_1\,.\,.\,\beta_{n-1} < \alpha_1\,\alpha_2\,.\,.\,\alpha_n. \qquad \ldots(9.16)$$

From the result of Section 4,

$$b_{2n-1} = \frac{D_{n-1}}{D_{n-2}} \qquad \ldots(4.3)$$

a repeated product would yield

$$b_1, b_3, b_5, \ldots b_{2n-3} = D_{n-2}, \qquad \ldots(9.17)$$

Theorem 15—The results of Theorem 4 could be interpreted by (9.18) and (9.19) below, which conform to those of Theorem 11.

b'_{2n-1} and b'_{2n} may be written as in (4.11) by multiplying the numerator and denominator in (4.10) by the missing terms as

$$b'_{2n-1} = \frac{b_1^2\;\; b_3^2\;.\;.\,b_{2n-3}^2}{b_0 b_1 b_2 b_3\;.\;.\;b_{2n-3}\;b_{2n-2}} = \frac{D_{n-2}^2}{c_0\,\beta_0\,\beta_1\,.\,.\,\beta_{n-2}} \qquad \ldots(9.18)$$

where we have used (9.17) above.

i. e. $b'_{2n-1} > 0$

also

$$b'_{2n} = \frac{c_0\,\beta_0\,\beta_1 \ldots \beta_{n-2}}{b_1^2\;\; b_3^2\;\; .\;.\,b_{2n-3}^2} \times \frac{1}{b_{2n-1}}$$

$$= \frac{c_0\,\beta_0\,\beta_1 \ldots\ldots \beta_{n-2}}{D_{n-2}^2} \times \left(\frac{D_{n-2}}{D_{n-1}}\right)$$

$$b'_{2n} = \frac{c_0\,\beta_0\,\beta_1\,.\,.\,\beta_{n-2}}{D_{n-1} \times D_{n-2}} \qquad \ldots(9.19)$$

obviously $b'_{2n-1} \,.\, b'_{2n} = \left(\frac{D_{n-2}}{D_{n-1}}\right) = \frac{1}{b_{2n-1}}$

which verifies the earlier result. In this respect results (9.18) and (9.19) are new.

Corollary 15.1—The product $b'_{2n-1} \cdot b'_{2n}$ is bounded from above, and b_{2n-1} are bounded from below.

From (9.10), changing n to $n - 1$, we have

$$D_{n-1} < \alpha_{n-1} D_{n-2}$$

$$\frac{1}{D_{n-2}} < \frac{\alpha_{n-1}}{D_{n-1}} .$$

Whence (9.19) becomes

$$b'_{2n} < c_0 \beta_0 \beta_1 \ldots . \beta_{n-2} . \frac{\alpha_{n-1}}{D^2_{n-1}} . \qquad \ldots(9.20)$$

Also from (9.6)

$$\frac{1}{\alpha_n D_{n-1}} < \frac{1}{\beta_{n-1}} \cdot \frac{1}{D_{n-2}}$$

or

$$\frac{1}{D_{n-1}} < \frac{\alpha_n}{\beta_{n-1}} \cdot \frac{1}{D_{n-2}}$$

this gives with (9.19), by replacing

$$\frac{1}{D_{n-1}}$$

the inequality

$$b'_{2n} < c_0 \beta_0 \beta_1 \ldots \beta_{n-2} \times \frac{\alpha_n}{\beta_{n-1}} \times \frac{1}{D^2_{n-2}} . \qquad \ldots(9.21)$$

From (9.18)

$$b'_{2n-1} = \frac{D^2_{n-2}}{c_0 \beta_0 \beta_1 \ldots \beta_{n-2}} . \qquad \ldots(9.18)$$

Multiplying (9.21) and (9.18), we get

$$b'_{2n-1} \cdot b'_{2n} < \frac{\alpha_n}{\beta_{n-1}} . \qquad \ldots(9.22)$$

But since we have

$$b'_{2n-1} \cdot b'_{2n} = \frac{1}{b_{2n-1}} = \frac{D_{n-2}}{D_{n-1}}$$

$$\frac{1}{b_{2n-1}} < \frac{\alpha_n}{\beta_{n-1}}$$

i. e. $\frac{\beta_{n-1}}{\alpha_n} < b_{2n-1}$. q. e.d. ...(9.23)

Theorem 16—$\{D_n\}$ is monotonically decreasing sequence of positive terms.

Since $|\alpha_l|, |\beta_l| < 1$ then obviously from (9.9).

$$D_n < D_{n-1}$$

or else

$$\frac{1}{D_n} > \frac{1}{D_{n-1}}.$$

So the determinants D_n are such that in magnitude the values of the reciprocals form increasing sequence

$$\frac{1}{D_n} > \frac{1}{D_{n-1}} > \frac{1}{D_{n-2}} \ldots > \frac{1}{D_0} \qquad \ldots(9.24)$$

where $D_0 = \alpha_0$.

Multiplying the inequalities (9.18) and (9.20)

$$b'_{2n-1} \cdot b'_{2n} < \frac{\alpha_{n-1}}{D^2_{n-1}} \times D^2_{n-2}.$$

Since $\alpha_l < 1$ and $\frac{1}{D_{n-1}} < \frac{1}{D_n}$

we get

$$b'_{2n-1} \cdot b'_{2n} < \frac{D^2_{n-2}}{D^2_n}. \qquad \ldots(9.25)$$

From a previous result we have

$$b'_{2n-1} \cdot b'_{2n} = \frac{1}{b_{2n-1}} = \left(\frac{D_{n-2}}{D_{n-1}}\right).$$

Thus

$$\frac{D_{n-2}}{D_{n-1}} < \frac{D_{n-2}^2}{D_{n-1}^2}$$

i.e.

$$\frac{D_{n-1}}{D_{n-2}} < 1. \qquad \ldots(9.26)$$

Thus $\{D_n\}$ forms decreasing sequence of positive terms $D_n > 0$ and would tend to its lower limit. Since a monotonically decreasing sequence of positive terms tends to zero, thus $D_n \to 0$. q.e.d

Theorem 17—In the tridiagonal positive definite representation D_n of K in H, the following inequality holds for the elements.

Product of the elements on the lower diagonal/ Product of the elements on the main diagonal $< | <$ Product of the elements in the upper diagonal/Product of the terms on the main diagonal.

As before. from the recurrence relations (9.4)

$$D_n = \alpha_n \; D_{n-1} - \beta_{n-1} \; D_{n-2}. \qquad \ldots(9.4)$$

Since $\{D_n\}$ is a decreasing sequence of positive terms, inequality (9.15) gives

$$\frac{\beta_0 \, \beta_1 \ldots \beta_{n-1}}{\alpha_1 \, \alpha_2 \ldots \alpha_n} < D_{n-1} < \ldots < D_0 = \alpha_0. \qquad \ldots(9.27)$$

Whence the first part of the inequality is established. To see the right side, since $0 < \alpha_i < 1$, we trivially get

$$1 < \frac{1}{\alpha_0 \alpha_1 \alpha_2 \ldots \alpha_n}. \qquad \ldots(9.28)$$

Combining (9.27) and (9.28) we establish the theorem. q.e.d.

Corollary 17.1—$\beta_0 \, \beta_1 \ldots \beta_{n-1} < \alpha_1 \, \alpha_2 \ldots \alpha_n \, D_{n-1}$. $\ldots(9.29)$

Using an earlier result (4.3)

$$b_{2n-1} = \frac{D_{n-1}}{D_{n-2}} \qquad \ldots(4.3)$$

we get $D_{n-1} = b_{2n-1} \; b_{2n-3} \ldots b_3 \; b_1$ $\ldots(9.30)$

Employing the fact that

$$b_{2n-1}\, b_{2n} = \beta_{n-1} \quad \ldots \qquad \ldots(4.5)$$

another variant of (9.29) is

$$b_2\, b_4 \ldots b_{2n} < \alpha_1\ \alpha_2 \ldots \alpha_n \qquad \ldots(9.31)$$

Which is true since by corollary 3.1

$b_{2n} + b_{2n+1} = \alpha_n$ and b_l are positive

i. e.

$$b_2 b_4 - - b_{2n} < (b_2 + b_3)(b_4 + b_5) - - - (b_{2n} + b_{2n+1}). \qquad \ldots(9.32)$$

Corollary 17.2 — $\beta_0\, \beta_1 \ldots \beta_{n-1} < \alpha_0\, \alpha_1^2\ \ \alpha_2^2 \ldots \alpha_{n-1}^2\ \ \alpha_n.$ $\qquad \ldots(9.33)$

From (9.4) by changing the index, we have

$$D_{n-1} < \alpha_{n-1}\, D_{n-2}. \qquad \ldots(9.34)$$

Multiplying such inequalities, we get

$$D_{n-1} < \alpha_{n-1}\, \alpha_{n-2} \ldots \alpha_1\, D_0. \qquad \ldots(9.35)$$

Substituting it in (9.29), we have (9.33).

The above result may as well be noted to be dimensionally compatible, since $\alpha_m = 0\,(\theta)$ and $\beta_m = 0\,(\theta^2)$, the inequality (9.33) above is of $0\,(\theta^{2n})$, where θ is defined in (8.3).

10 Truncation Error

We can now compute the truncation error when the *S*-fraction in (4.1) is truncated at the $2n$th denominator given by the recurrence relations (5.1).

Let $E_{2n}(\xi)$ denote the truncation error, then in the view of the positive nature of the various qualities involved, the classical inequality yields:

$$\frac{1}{2Q_{2n+1}^2} < E_{2n}(\xi) < \frac{1}{Q_{2n}^2}. \qquad \ldots(10.1)$$

From the recurrence relation

$$Q_{2n} = b'_{2n}\, \xi\, Q_{2n-1} + Q_{2n-2} \qquad \ldots(10.2)$$

and the fact that Q_{2n} is an increasing sequence of positive terms, the arithmetic-geometric property gives

$$Q_{2n}^2 > 4b'_{2n}\,\xi Q_{2n-1}\,Q_{2n-2}. \qquad \ldots(10.3)$$

Substituting the value of b'_{2n} from (9.19) and using (5.5) for the changed index

$$X_{2n-1} = Q_{2n-1}\,Q_{2n-2} \qquad \ldots(10.4)$$

the right inequality in (10.1) gives

$$E_{2n}(\xi) < \frac{1}{4\xi\, c_0\, \beta_0 \ldots \beta_{n-2}} \cdot \frac{D_{n-1} \times D_{n-2}}{X_{2n-1}}. \qquad \ldots(10.5)$$

In order to see the left side of the inequality (10.1), we employ

$$Q_{2n+1} = b'_{2n+1}\,\xi\, Q_{2n} + Q_{2n-1} \qquad \ldots(10.6)$$

and the fact that

$$Q_{2n+1} > Q_{2n} > Q_{2n-1}$$

which gives

$$X_{2n+1} = Q_{2n+1}\,Q_{2n} > Q_{2n+1}.\ Q_{2n\ 1}. \qquad \ldots(10.7)$$

Consequently,

$$Q_{2n+1}^2 < \left(b'_{2n+1}\,\xi + 1 \right) X_{2n+1}$$

whence we get

$$\frac{1}{2\left(b'_{2n+1}\,\xi + 1 \right)} \cdot \frac{1}{X_{2n+1}} < E_{2n}(\xi) \qquad \ldots(10.8)$$

where

$$b'_{2n+1} = \frac{D_{n-1}^2}{c_0\beta_0\beta_1 \ldots \beta_{n-1}}.$$

Another variation of the above from (10.6) is that

$$Q_{2n+1}^2 < \frac{2\xi X_{2n+1} D_{n-1}^2}{c_0\beta_0\beta_1 \ldots \beta_{n-1}} + Q_{2n-1}^2$$

$$< \frac{4\xi X_{2n+1}}{c_0\beta_0\beta_1 \ldots \beta_{n-1}} \cdot D_{n-1}^2$$

whence,

$$\frac{1}{2Q_{2n+1}^2} > \frac{c_0\beta_0\beta_1 \ldots \beta_{n-1}}{8\xi X_{2n+1}} \cdot \frac{1}{D_{n-1}^2}. \qquad \ldots(10.9)$$

Using the fact that $D_{n-1} < \alpha_{n-1} D_{n-2}$, the left inequality in (10.1) becomes

$$\frac{c_0\beta_0\beta_1 \ldots \beta_{n-1}}{8\xi \alpha_{n-1} D_{n-1} D_{n-2}} \cdot \frac{1}{X_{2n+1}} < E_{2n}(\xi). \qquad \ldots(10.10)$$

By changing the index in (5.11) we get

$$\frac{1}{X_{2n+2}} \leqslant \frac{1}{\left[1 + 2\xi\left(\frac{D_{n-1}}{D_n}\right)^{1/2}\right]} \cdot \frac{1}{X_{2n}} \qquad \ldots(10.11)$$

and in particular,

$$\frac{1}{X_2} < \frac{\alpha_0 c_0}{(\alpha_0 + \xi^2)\xi}. \qquad \ldots(10.12)$$

Let

$$M = \frac{D_{n-1} \cdot D_{n-2}}{4c_0\beta_0 \ldots \beta_{n-2}} \qquad \ldots(10.13)$$

and since $\xi^2 = \lambda$, (10.5) gives the upper bound as

$$E_{2n}(\xi) < \frac{M}{\lambda}\left(\frac{\alpha_0 c_0}{\alpha_0 + \lambda}\right)^{\frac{1}{2}}. \qquad \ldots(10.14)$$

Using (10.13) in (10.5) and (10.10), we get the analogue of the classical inequality (10.1) in terms of the elements of the continued fraction as

$$\frac{\xi}{32M}\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right)\frac{1}{X_{2n+1}} < E_{2n}(\xi) < \frac{M}{\xi} \cdot \frac{1}{X_{2n-1}} \qquad (10.15)$$

Thus we may state the following :

Theorem 18—The truncation error $E_{2n}(\xi)$ is bounded, the bounds are given in terms of D_n and Q_{2n} by (10.15).

From (10.6), changing the index and multiplying with Q_{2n+1}, we get the recurrence relation

$$X_{2n+2} = b'_{2n+2}\ \xi\ Q^2_{2n+1} + X_{2n+1}. \qquad \ldots(10.16)$$

A repeated application of the above yields

$$X_{2n+2} = \xi \sum_{i=0}^{2n+1} b'_{i+1}\ Q_i^2\ . \qquad \ldots(10.17)$$

Since by (4.11) $\left\{b'_i\right\}$ are positive, $\{X_{2n}\}$ form an increasing sequence of partial sums [cf. Theorem 5 and Corollary 5.1].

Since all the quantities in (10.2) are positive, we have

$$Q_{2n} > b'_{2n}\ \xi\ Q_{2n-1} \qquad \ldots(10.18)$$

using the value af b'_{2n} from (9.19) and the fact that $D_n < 1$, we get :

$$\frac{1}{b'_{2n}} < \frac{1}{c_0\beta_0\beta_1 \cdots \beta_{n-2}} \qquad \ldots(10.19)$$

whence we have,

Theorem 19 (a)—The successive denominators $\{Q_{2n}\}$ form an increasing sequence of positive terms such that

$$\frac{1}{Q_{2n}} < \frac{1}{c_0\xi\beta_0\beta_1 \cdots \beta_{2n-1}} \cdot \frac{1}{Q_{2n-1}} \qquad \ldots(10.20)$$

The above result is refined is the sense that in replacing the b'_{2n}, we use (9.19) and the fact that $D_{n-1} < \alpha_{n-1}\ D_{n-2}$ which gives

$$\frac{1}{Q_{2n}} < \frac{\alpha_{n-1}}{c_0\xi\beta_0\beta_1 \cdots \beta_{n-2}} D^2_{n-2} \cdot \frac{1}{Q_{2n-1}}\ . \qquad \ldots(10.21)$$

Employing the bound as in (7.7) for

$$D_0^{(n-2)} \text{ i. e. } D_{n-2} < \frac{1+\epsilon}{\lambda_0^{n-1}}. \qquad \ldots (10.22)$$

The above result may be summarized as

Theorem 19(b)—$\{Q_{2n}\}$ are such that

$$Q_{2n-1} < \frac{\alpha_n \; 1}{c_0 \xi \beta_0 \beta_1 \; . \; . \; \beta_{n-1}} \left(\frac{1+\epsilon}{\lambda_0^{n-1}} \right)^2 Q_{2n} \qquad \ldots (10.23)$$

where λ_0^{-1} corresponds to the first even value of K.

Alternately, employing the fact that $D_n < \alpha_n \; D_{n-1}$ the product $D_{n-1} . \; D_{n-2}$ is suitably replaced to give the following :

Theorem 19(c)—$\{Q_{2n}\}$ are such that

$$Q_{2n-1} < \frac{\alpha_{n-1} \; \gamma_{n-2}^2 \; \ldots \; \alpha_0^2}{c_0 \xi \beta_0 \beta_1 \; \ldots \; . \; \beta_{n-1}} Q_{2n}. \qquad \ldots (10.24)$$

Since the structure of $0 < \alpha_i < 1, \; 0 < \beta_i < 1$ is such that $\beta_{i+1} < \alpha_i$, the inequality (10.24) could be written in a more symmetric from

$$\textit{Corollary } 19.1 — \; Q_{2n-1} < \frac{1}{c_0 \xi} \prod_{i=0}^{n-1} \left(\frac{\alpha_i}{\beta_i} \right) . \; Q_{2n}. \qquad \ldots (10.25)$$

From the recurrence relations (10.2), neglecting the positive term on the right, we see that $Q_{2n} > Q_{2n-2}$.

Thus the denominators of even approximants form an increasing sequence. Moreover, from (10.18) three possiblities arise, either $b'_{2n} \; \xi \lesseqgtr 1$. Since $b'_{2n} \to 0$. and ξ being the parameter connected with the reciprocal of the eigen value, the only possibility that conforms to the situation is that

$$b'_{2n} \; \xi < 1.$$

The other two cases $b'_{2n} \; \xi \geqslant 1$. trivially establish that $\{Q_{2n}\}$ is an increasing sequence. Consequently, the radius of convergence for $P(\lambda)$ would be determined by

$$\lambda < \frac{1}{b'^2_{2n}} \quad \ldots(10.26)$$

Using an estimate of b'_{2n} as in (10.23), the above inequality may be normalized as :

Theorem 20—If λ_0^{-1} is the first eigen value of K, then the series $P(\lambda)$ will converge for

$$\xi = \sqrt{\lambda} < \frac{\alpha_{n-1}}{c_0\beta_0\beta_1 \ldots \beta_{n-1}} \left(\frac{1+\epsilon}{\lambda_0^{n-1}} \right)^2 \quad \ldots(10.27)$$

where ϵ is a small quantity independent of λ_0.

Now we are in a position to establish that $\{E_{2n}\}$ form a decreasing sequence as opposed to $\{X_{2n}\}$ and $\{Q_{2n}\}$ which have been shown to form increasing sequences.

Let the continued fraction (4.8) have the value $S(\xi)$. Then

$$E_{2n}(\xi) = S(\xi) - \frac{P_{2n}(\xi)}{Q_{2n}(\xi)} \quad \ldots(10.28)$$

and

$$E_{2n-1}(\xi) = S(\xi) - \frac{P_{2n-1}(\xi)}{Q_{2n-1}(\xi)} . \quad \ldots(10.29)$$

Employing (10.1) the following is obvious

$$\frac{1}{4Q^2_{2n+1} \quad Q^2_{2n}} < E_{2n} E_{2n-1} < \frac{1}{Q^2_{2n} \quad Q^2_{2n\ 1}} \quad \ldots(10.30)$$

while using (6.6) we get

$$E_{2n-1} - E_{2n} = \frac{1}{\xi X_{2n}} . \quad \ldots(10.31)$$

Since the right hand side above is positive we conclude that

$$E_{2n} < E_{2n-1} \quad \ldots(10.32)$$

using this in right side of (10.30) we have

$$E_{2n} < \frac{1}{X_{2n}} \quad \ldots(10.33)$$

which with the help of (10.31) yields

$$E_{2n} < \frac{\xi}{1+\xi} E_{2n-1}. \qquad \ldots(10.34)$$

This establishes that the sequence experiences a geometric decrease with a factor

$$\left|\frac{\xi}{1+\xi}\right| < 1.$$

A repeated application of (10.34) helps state the following :

Theorem 21—$\{E_{2n}\}$ forms a decreasing sequence such that

$$E_{2n} < \left(\frac{\xi}{1+\xi}\right)^{2n-2} E_2 \qquad \ldots(10.35)$$

where

$$E_2 < \frac{c_0 \alpha_0}{\xi(\alpha_0 + \xi^2)} \text{ and } E_0 = 0. \qquad \ldots(10.36)$$

The above can easily be seen with the help of (10.12), (10.33) and (10.34). q.e.d.

To conclude, we recapitulate that the Neumann series solution of functional equation (2.1) in the Hilbert Space H endowed with Dirichlet norm, gives a polynomial in λ with Schwarz's constants c_i as coefficients. The decomposition of this polynomial into an S-fraction generates for its elements α_i, β_i that are the coefficients of Lanczos minimized iteration. Besides stating various properties of the continued fractions, we have noted the bounds for the truncation error E_{2n} and shown that it forms a decreasing sequence.

In the foregoing discussion, symmetric nature of K in H, positive definite character of tridiagonal matrices $\{D_n\}$ and properties of α_i, β_i constitute the essential ingredients.

We conjecture that this analysis can be extended to a system of functional equations or to the multi-parameter eigenvalue problem. This, however, would only be possible if the tridiagonal matrix with the above properties could be factored into similar matrices.

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