APPLICATION OF FUNCTIONAL ANALYSIS TO SOLVE CERTAIN CLASS OF CONSTRAINED TIME OPTIMAL CONTROL PROBLEMS

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The minimum time optimal control problem of linear systems under amplitude constraints can be easily solved by Pontryagin's maximum principle. However if there is an energy constraint in addition to the amplitude constraint, it becomes rather difficult to solve the problem by Pontryagin's maximum principle. In this paper we have proposed a functional Analytic approach to tackle such problems.

1. Introduction

The control systems, which can be characterised by the following vector matrix differential equation:

\[ \frac{dx}{dt} = A(t)X(t) + B(t)U(t) \] ...(1)

where \( X(t) \) is an \( n \) vector, representing the instantaneous state of the system, \( u(t) \) is an \( r \)-vector \( (r \leq n) \) representing the control input to the system, \( A(t) \) is \((n \times n)\) matrix and \( B(t) \) is an \((n \times r)\) matrix has received considerable attention in the literature. The solution of the above equation can be expressed in the following integral forms:

\[ X(t) = \phi(t, t_0)X(t_0) + \int_{t_0}^{t} \phi(t, s)B(s)U(s)ds \] ...(2)
where \( \phi \left( t, t_0 \right) \) is the fundamental matrix of the system (1), and \( x \left( t_0 \right) \), the initial state of the system at time \( t = t_0 \). The minimum time control problem, is to find the optimal control \( u \left( t \right) \) belonging to the admissible set, which will derive the systems from a given initial state \( x \left( t_0 \right) \) at \( t = t_0 \), to the desired state \( x_1 \) in minimum time \( t \) i.e. \( x \left( t \right) = x_1 \). Now (2) can be written as \( X \left( t \right) = \phi \left( t, t_0 \right) X \left( t_0 \right) = \int_{t_0}^{t} \phi \left( t, s \right) B \left( s \right) U \left( s \right) ds \).

Put \( X \left( t \right) - \phi \left( t, t_0 \right) X \left( t_0 \right) = \xi \).

Expression (2) can be written as \( \xi = T_t u \)

where

\[
T_t u = \int_{t_0}^{t} \phi \left( t, s \right) B \left( s \right) U \left( s \right) ds.
\]

Thus without any loss of generality one can consider the problems of finding the optimal \( u \) to drive the system from the origin to any point \( \xi \) in minimum time \( t \).

The above problem can be considered as a mapping from some space to which \( u \) belongs to some other space to which \( \xi \) belongs. In the light of the above we can consider following general problem:

Let \( B_t \) be a Banach space depending on the parameter \( t \) and \( D \) be also a Banach space. Let \( T_t \) be a bounded linear transformation depending on the parameter mapping \( B_t \) on-to \( D \). Let \( \xi \in D \). The problem is to find the optimal control \( u \left( . \right) \in B_t \) to reach \( \xi \) from the origin in minimum time \( t \) under the constraint \( \| u \left( . \right) \| \leq 1 \) where \( \| u \left( . \right) \| \) denotes the norm of \( u \left( . \right) \) in \( B_t \).

In many practical problems in engineering and technology, it turns out that one can identify a suitable function space to which the control function belongs, such that constraint turns out precisely to be the constraint on the norm of \( u \) in that above space (Ref. 9).

Chaudhuri and Mukherjee (Ref. 2, 3), considered the minimum control for systems which can be represented as a mapping from a Banach space of control functions to another Banach space of the state of the control systems. The Banach space of control functions were either a Reflexive space or a conjugate of some other Banach space.

**Problem Statement**

In this paper we shall consider the problem where the constraints on the control function are given as follows:

\[
| u \left( \tau \right) | \leq N, \left( \int_{t_0}^{t} | u \left( \tau \right) |^2 d\tau \right)^{1/2} \leq M
\]
$M$ and $N$ being to positive constraints. The problem is to find the optimal control function $u$ which will drive the origin (initial state) to $\xi$ (desired state) in minimum time $t$, satisfying the above constraints.

For the sake of completeness, we shall now give certain definitions, theorems and lemmas.

**Definitions**—The set of all points $\xi \in D$, such that $T_t u = \xi$ for some $u \subseteq U_t \subseteq B_t$ will be called the Reachable set and will be denoted by $C(t)$, where $U_t$ is the unit ball in $B_t$, for some given time $t$.

In following theorems $B_t$, $T_t$, $D$, will mean the same as defined earlier, until they are specifically defined.

**Theorem 1**—If $B_t$ and $D$ be the conjugate spaces of the normed linear spaces $X_t$ and $Y$ respectively and $T_t$ is the adjoint of some bounded linear transformation $S$, mapping $Y$ one to one and onto a closed subspace of $X_t$, then $C(t)$ is closed.

**Proof:** By Alaoglu's theorem the unit ball in $X_t^*$ is weak* compact. Also, both $X_t^*$ and $D$ are equipped with their weak* topologies. Again, as $T_t$ is adjoint to $S$, $T_t$ will also be onto and remains continuous with respect to weak* topologies of $X_t^*$ and $D$. Consequently, the unit ball of $X_t^*$ will be mapped onto a weak* compact subset of $D$. Hence $C(t)$ is weak* closed, and therefore weakly closed and hence norm closed in $D$.

**Note:** Let $X$ be a normed linear space and $X^*$ be its conjugate. Hahn-Banach theorem assures that, for each $x (\neq \theta) \in X$ there exists an $f \in X^*$ which satisfies the condition $f(x) = \|x\|$ and $\|f\| = 1$. Such an $f$ will be called an extremal of $x$. ($X^*$ denotes the conjugate space of the normed linear space $X$).

**Note:** The Reachable set is also convex body, symmetric with respect to the origin of $D$ (Ref. 3).

**Theorem 2**—Let $B_t$ be the conjugate space of the normed linear space $X_t$ and $D$ is the conjugate of some normed linear space $Y$. Let $\xi \in \delta C(t)$, where $\delta C(t)$ denotes the boundary of $C(t)$ for some given time $t$. Then there exists at least one $u_t(\xi) \in U_t \subseteq B_t$ which will transfer the system from origin to $\xi \in \delta C(t)$ in minimum time $t$, where $T_t$ is as in Theorem 1.

**Proof:** Here, $S : Y \rightarrow X_t$, $S^* : X_t^* \rightarrow Y^*$ i.e. $S^* : B_t \rightarrow D : S^* = T_t$. If $\phi \in D^*$, then $T_t^* \phi \in B_t^*$, $\therefore T_t^* \phi \in B_t^{**}$ where $T_t^* \phi$ denotes the extremal of $T_t^* \phi$; where $\|T_t^* \phi\| = 1$. 

Since, $T_i$ is onto, (ref. 5, 9), hence by the open mapping theorem, the optimal control $u_\xi \in U_t \subset B_i$. Hence there exists at least one $u_\xi (t) = T^*_i \phi \in U_t \subset B_i^* \subset B_i^{**}$. Now if $t^*$ is the minimum time to reach $\xi$, then $\xi \in \delta C(t^*)$. Let $\phi_\xi \in D^*$ be the supporting hyperplane at $\delta C(t^*)$ at $\xi$. Let $u_\phi$ be the optimal control to reach $\xi$ in minimum time $t^*$ — then $u_\phi = T^*_i \phi$, $\|u_\phi\| = 1$ (see ref 2, 3). See Burns’ et al. for determining $\phi_\xi$ and $t^*$ for a given $\xi$.

**Remark 1**: If $D$ in finite dimensional, then $S$ always exist. We state the following lemmas which can be easily proved.

**Lemma 1**—Let $X$ be a normed linear space. If $\rho_1(x)$ and $\rho_2(x)$ are the seminorm and norms respectively in $X$, then, $\text{Max} \{\rho_1(x), \rho_2(x)\}$ is a norm in $X$, where $x \in X$.

**Corollary**—Evidently $\text{Max} \{\rho_1(x), \rho_2(x)\}$ is a norm, where each of $\rho_1(x), \rho_2(x)$ is a norm.

**Lemma 2**—$\|u\|_{\infty, N, t^{2M}} = \text{Max} \left\{ \text{ess sup}_{0 < t \leq \tau \leq t} \frac{|u(\tau)|}{N}, \frac{1}{M} \left\{ \int_0^t |u(\tau)|^2 \, d\tau \right\}^{1/2} \right\}$

is equivalent to $\|u\|_{\infty} = \text{ess sup}_{0 < \tau < t} |u(\tau)|$ which is a norm on $L_\infty(0, t)$.

**Proof**: We have

\[
\frac{1}{M} \left\{ \int_0^t |u(\tau)|^2 \, d\tau \right\}^{1/2} \leq \frac{1}{M} \text{ess sup}_{0 < \tau < t} |u(\tau)| \cdot \sqrt{t}
\]

\[
= \frac{N}{M} \left( \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} \right) \sqrt{t}.
\]

We shall consider two cases, case (i) and case (ii), and two subcases of case (ii).

**Case (i)**—If $t \leq \frac{M^2}{N^2}$, \( \frac{1}{M} \left( \int_0^t |u(\tau)|^2 \, d\tau \right)^{1/2} \leq \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} \)

\[= \|u\|_{\infty, N, t^{2M}} = \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} = \|u\|_{\infty, N} \text{ for } t \leq \frac{M^2}{N^2}. \quad \text{...}(3)\]

Hence $\|u\|_{\infty, N, t^{2M}}$ is equivalent to $\|u\|_{\infty}$ for $t \leq \frac{M^2}{N^2}$.
Case (ii) \(-t > \frac{M^2}{N^2}\). There will be two subcases:

(a) \(\text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} = \frac{1}{M} \left( \int_0^t |u(\tau)|^2 \, d\tau \right)^{1/2}\)

on a set of finite measure.

(b) \(\text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} < \frac{1}{M} \left( \int_0^t |u(\tau)|^2 \, d\tau \right)^{1/2}\) almost everywhere.

We shall make use of the following notations:

\[\|u\|_{\infty, N} = \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N}, \quad \|u\|_{2, M} = \frac{1}{M} \left( \int_0^t |u(\tau)|^2 \, d\tau \right)^{1/2}.\]

Obviously \(\|u\|_{\infty, N}\) and \(\|u\|_{2, M}\) are norms and they are equivalent to \(\|u\|_{\infty}\) and \(\|u\|_2\), respectively.

In case (ii) (a) \(\|u\|_{\infty, N, 2, M} = \|u\|_{\infty, N} = \|u\|_{2, M}\)

\[\therefore \|u\|_{\infty, N, 2, M} \leq \|u\|_{\infty, N} \leq \|u\|_{\infty, N, 2, M} \quad \ldots(4)\]

In case (ii) (b) \(\|u\|_{\infty, N, 2, M} = \|u\|_{2, M} \leq \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{M} \cdot \sqrt{t}\)

or

\[\frac{M}{N\sqrt{t}} \|u\|_{\infty, N, 2, M} \leq \text{ess sup}_{0 < \tau < t} \frac{|u(\tau)|}{N} = \|u\|_{\infty, N} \leq \|u\|_{2, M}\]

combining (3), (4) and (5) we obtain

\[\text{Max} \left\{1, \frac{M}{N\sqrt{t}} \right\} \|u\|_{\infty, N, 2, M} \leq \|u\|_{\infty, N} \leq \|u\|_{\infty, N, 2, M}. \quad \ldots(6)\]

But \(\|u\|_{\infty, N}\) is obviously equivalent to \(\|u\|_{\infty}\). Hence from (6) \(\|u\|_{\infty, N, 2, M}\) is equivalent to \(\|u\|_{\infty}\). Hence the proof.
**Definition**—We define \( L_{\infty,N_{1},M} \) to be the space of all essentially bounded functions \( u \), equipped with the norm \( \| u \|_{\infty,N_{1},M} \).

**Definition**—We define \( L_{\infty,N} \) to be the space of all essentially bounded functions \( u \), equipped with the norm \( \| u \|_{\infty,N} = \text{ess sup}_{0 \leq \tau \leq t} \frac{\left| u(\tau) \right|}{N} \).

**Definition**—The space \( L_{2,M} \) consists of all square integrable functions \( u \), equipped with the norm \( \| u \|_{2,M} = \frac{1}{M} \left( \int_{0}^{t} \frac{\left| u(\tau) \right|^2}{d\tau} \right)^{1/2} \).

**Note:** Evidently \( \| u \|_{\infty,N} \) and \( \| u \|_{2,M} \), are equivalent to \( \| u \|_{\infty} \) and \( \| u \|_{2} \) respectively and hence the spaces \( L_{\infty,N} \) and \( L_{2,M} \) are complete with respect to their respective norms \( \| u \|_{\infty,N} \) and \( \| u \|_{2,M} \).

Consider a system described by (1) where \( u(t) \) is a scalar control. Assume that at \( t = 0 \) the state of the system is given by \( x(0) \). It is required to find \( u(t) \) which will bring the system from the initial state \( x(0) \) to the origin of the state space in the least time under the constraint \( \left| u(\tau) \right| \leq N \left( \int_{0}^{t} \frac{\left| u(\tau) \right|^2}{d\tau} \right)^{1/2} \leq M \).

The above constraints can be expressed in the following alternative forms:

\[
J(u) = \text{Max} \left\{ \text{ess sup}_{0 \leq \tau \leq t} \frac{|u(\tau)|}{N}, \frac{1}{M} \left( \int_{0}^{t} \left| u(\tau) \right|^2 \ d\tau \right)^{1/2} \right\}.
\]

From Lemma 1, it follows that \( J(u) \) is a norm in \( L_{\infty,N_{1},M} \).

Now \( L_{\infty,N_{1},M} \) can be considered as the conjugate of the space \( L_{1,N_{1},M} \) i.e. \( L_{1,N_{1},M} = L_{\infty,N_{1},M} \) where * denotes the conjugate of the corresponding spaces. Here \( T_t : L_{\infty,N_{1},M} \to R^n \) where \( R^n \) denotes the \( n \)-dimensional Euclidean space. In the finite dimensional case it can be easily shown that \( T_t^* = S \) is one to one and onto a closed subspace of \( L_{1,N_{1},M} \), where \( S : R^n \to L_{1,N_{1},M} \). By Theorem 1 one can easily verify that the corresponding Reachable set is closed. Also, By Theorem 2, it follows that there exists an optimal control \( u_\phi \).

**The Form of the Optimal Control**

The problem is to find \( u \) which will maximize \( \langle u, T_t^* \phi \rangle \).
Under the constraint \(| u (\tau) | \leq N, \frac{1}{M} \left( \int_{0}^{t} | u (\tau) |^2 d\tau \right)^{1/2} \leq 1.\)

**Case (I)**—If \( t \leq \frac{M^2}{N^2} \) then \( \|u\|_{\infty;N_{12}, M} = \text{ess sup} \frac{|u (\tau)|}{N} = 1 \)

\[
\therefore \text{ess sup} | u (\tau) | = N.
\]

Now, the optimal \( u \) must satisfy the condition\(^{29-31-9}\) \(< u, T^*_t \phi > = \|T^*_t \phi\|_{1,N} \) and \( \|u\|_{\infty;N} = 1. \) So the problem is to find a \( u \), which will maximize

\[
<u, T^*_t \phi > = \int_{0}^{t} u (\tau) \left( T^*_t \phi \right) (\tau) d\tau
\]

subject to \( \text{ess sup} \frac{|u (\tau)|}{N} = N. \)

Evidently the optimal \( u (t) \) will be given by \( u_{\phi} (\tau) = N \) \( \text{sign} \left[ T^*_t \phi (\tau) \right], 0 \leq \tau \leq t \)

and \(< u, T^*_t \phi > = N \int_{0}^{t} \left| \left( T^*_t \phi \right) (\tau) \right| d\tau. \)

It can be easily verified that \( \|T^*_t \phi\|_{1,N} = N \int_{0}^{t} | T^*_t \phi (\tau) | d\tau. \)

**Case (II) (a)**—\( \|u\|_{\infty;N_{12}, M} = \text{ess sup} \frac{|u (\tau)|}{N} = \frac{1}{M} \left( \int_{0}^{t} | u (\tau) |^2 d\tau \right)^{1/2} \leq 1. \)

Hence \( \text{ess sup} | u (\tau) | = N \) and \( \int_{0}^{t} | u (\tau) |^2 d\tau = M^2. \)

Consequently, one has to find that \( u (\tau) \) which will maximize \( < u, T^*_t \phi > = \int_{0}^{t} u (\tau) \)
\[
\begin{align*}
&\left( T_t^* \phi \right) (\tau) d \tau. \quad \text{Let } E = \{ t : u (\tau) = N \} \text{ and } E_c = \{ t : u (\tau) < N \} \\
&\therefore \int_{E}^{t} u (\tau) \left( T_t^* \phi \right) (\tau) d \tau = \int_{E}^{t} u (\tau) \left( T_t^* \phi \right) (\tau) d \tau \\
&\quad + \int_{E_c}^{t} u (\tau) \left( T_t^* \phi \right) (\tau) d \tau.
\end{align*}
\]

Now \( \int_{E}^{t} u (\tau) \left( T_t^* \phi \right) (\tau) d \tau \) will be maximized if \( u (\tau) = N \) \( \text{sign} \left[ \left( T_t^* \phi \right) (\tau) \right] \), \( \tau \in E \).

Again \( \int_{E}^{t} |u (\tau)|^2 d \tau = M^2 \) i.e. \( \int_{E}^{t} u (\tau) |u (\tau)|^2 d \tau + \int_{E_c}^{t} u (\tau) |u (\tau)|^2 d \tau = M^2 \)

or \( \int_{E_c}^{t} u (\tau) |u (\tau)|^2 d \tau = M^2 - N^2 m(E) \)

where \( m(E) \) denotes the measure of the set \( E \). So \( \int_{E_c}^{t} u (\tau) \left( T_t^* \phi \right) (\tau) d \tau \) will be maximized under the constraint (A), if we take \( u (\tau) = \alpha \left( T_t^* \phi \right) (\tau) \) where \( \alpha \) is a positive constant. Substituting \( u = \alpha \left( T_t^* \phi \right) (\tau) \) in (A), we have

\[
\alpha^2 \int_{E_c}^{t} \left( |T_t^* \phi| (\tau) \right)^2 d \tau = M^2 - N^2 m(E)
\]

where

\[
\alpha = \frac{\sqrt{M^2 - N^2 m(E)}}{\sqrt{\int_{E_c}^{t} \left| T_t^* \phi (\tau) \right|^2 d \tau}}
\]

\[
\text{Max } < u, T_t^* \phi > = N \int_{E}^{t} \left( T_t^* \phi \right) (\tau) d \tau + \sqrt{M^2 - N^2 m(E)}
\]

\[
\frac{\sqrt{\int_{E_c}^{t} \left| T_t^* \phi (\tau) \right|^2 d \tau}}{\sqrt{\int_{E_c}^{t} \left| T_t^* \phi (\tau) \right|^2 d \tau}}.
\]
It can be easily verified that
\[
\| T_i^* \phi \|_{1,N^{25},M} = N \int_{E} \left| T_i^* \phi (\tau) \right| d\tau + \sqrt{M^2 - N^2 m(E)}
\]
\[
\sqrt{\int_{E_c} \left| T_i^* \phi (\tau) \right|^2 d\tau}
\]
from the above it follows that
\[
u_{\phi} (\tau) = N \text{ Sign} \left( \alpha \left( T_i^* \phi \right) (\tau) \right),
\]
\[
\tau \in E = \{ t : \left| \alpha \left( T_i^* \phi \right) (\tau) \right| > N \} = \alpha \left( T_i^* \phi \right) (\tau), \tau \in E_c = \{ t : \left| \alpha \left( T_i^* \phi \right) (\tau) \right| \leq N \}.
\]

**Case (II) (b)** \( \| u \|_{\infty,N^{25},M} = \frac{1}{M} \left( \int_0^t \left| u (\tau) \right|^2 d\tau \right)^{1/2} = 1 \)
or,
\[
\int_0^t \left| u (\tau) \right|^2 d\tau = M^2.
\]

Now, the problem becomes, find \( u \) which will maximize \( \int_0^t u (\tau) \left( T_i^* \phi \right) (\tau) d\tau \)
under the constraint (B).

Obviously \( u_{\phi} = \alpha T_i^* \phi \), such that \( \alpha^2 \int_0^t \left| T_i^* \phi (\tau) \right|^2 d\tau = M^2 \)
i.e.
\[
\alpha = \frac{M}{\sqrt{\int_0^t \left| T_i^* \phi (\tau) \right|^2 d\tau}}.
\]
Thus

\[ u_\phi (\tau) = \alpha \frac{M \, T_i^* \phi (\tau)}{\sqrt{\int_{0}^{t} |T_i^* \phi (\tau)|^2 \, d\tau}} \]

and

\[ \int_{0}^{t} u (\tau) \left( T_i^* \phi (\tau) \right) d\tau = \alpha \int_{0}^{t} |T_i^* \phi (\tau)|^2 d\tau \]

\[ = M \left( \int_{0}^{t} |T_i^* \phi (\tau)|^2 d\tau \right)^{1/2} \]

\[ = \| T_i^* \phi \|_{1, N/2, M}. \]

**Example**—In positional control by separately, excited D. C. motor the load is moved to the desired position by supplying armature current to the D. C. motor.

The differential equation of the system is

\[ \frac{dwm}{dt} = - \frac{B_{me}}{J_{me}} \omega_m (t) + \frac{R_i}{J_{me}} i_a (t) \]

\[ \frac{d\theta_m}{dt} = Wm \]

where \( J_{me} \) is the effective inertia of the motor; \( B_{me} \) the effective viscous frictional coefficient; \( \theta_m \) the angular position of the motor; \( R_i \) the torque constant; and \( i_a \) the armature current and is the control. In such system the amplitude of the current should be limited. Also it is desirable the power-loss due to dissipation in the motor should not be permitted to exceed a given bound—which can be expressed as

\[ \left\{ \int_{0}^{t} |i_a (\tau)|^2 d\tau \right\}^{1/2} \ll M^2. \]

Thus the problem is to find the control \( i_a (t) \) so that system can be transferred from any position to any other position in minimum time. For a particular controller, \( B_{me} = 10^{-3} \text{ lb ft/sec} \), \( J_{me} = 2.10^{-3} \text{ lb ft/sec}^2 \), \( R_i = \text{ lb ft/amp} \).
Hence the system can be written as
\[
\frac{d}{dt} \begin{bmatrix} \theta_m \\ w_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} \theta_m \\ w_m \end{bmatrix} + \begin{bmatrix} 0 \\ 250 \end{bmatrix} i_a.
\] ... (C)

The solution of (C) can be written as
\[
e^{-At} \begin{bmatrix} \theta_m (t) \\ w_m (t) \end{bmatrix} - \begin{bmatrix} \theta_m (0) \\ w_m (0) \end{bmatrix} = \int_0^t e^{-As} B U(s) d s
\] ... (D)

where
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 250 \end{bmatrix}.
\]

If the system reaches the null state at time \( t \), the equation (D) reduces to
\[
-\begin{bmatrix} \theta_m (0) \\ w_m (0) \end{bmatrix} = \int_0^t \begin{bmatrix} 250S \\ 250 (1 - 0.5s) \end{bmatrix} i_a ds.
\]

Putting
\[
-\begin{bmatrix} \theta_m (0) \\ w_m (0) \end{bmatrix} = \xi,
\]

\[
\therefore \xi = \int_0^t \begin{bmatrix} 250S \\ 250 (1 - 0.5s) \end{bmatrix} i_a ds = T_i u.
\]

\[
\therefore T_i^* \phi_\xi = 250 \begin{bmatrix} t, 1 - 0.5t \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 250 (t \phi_1 + (1 - 0.5t) \phi_2)
\]

\[
\therefore u_\phi (t) = N Sign \left[ \alpha (250 t \phi_1 + 250 (1 - 0.5t) \phi_2) \right], \; t \in E
\]

\[
= \alpha [250 t \phi_1 + 250 (1 - 0.5t) \phi_2], \; t \in E_e
\]

where
\[
\alpha = \frac{\sqrt{M^2 - N^2}}{\sqrt{\int_{E_e} (250)^2 [t \phi_1 + (1 - 0.5t) \phi_2]^2 dt}}
\]

\[
E = \{ t : \alpha [250 t \phi_1 + 250 (1 - 0.5t) \phi_2] > N \}
\]

\[
E_e = \{ t : \alpha [250 t \phi_1 + 250 (1 - 0.5t) \phi_2] \leq N \}.
\]
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