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APPLICATION OF FUNCTIONAL ANALYSIS TO SOLVE CERTAIN CLASS OF CONSTRAINED TIME OPTIMAL CONTROL PROBLEMS

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The minimum time optimal control problem of linear systems under amplitude constraints can be easily solved by Pontryagin's maximum principle. However if there is an energy constraint in addition to the amplitude constraint, it becomes rather difficult to solve the problem by Pontryagin's maximum principle. In this paper we have proposed a functional Analytic approach to tackle such problems.

1. INTRODUCTION

The control systems, which can be characterised by the following vector matrix differential equation:

$$\frac{dx}{dt} = A(t)\,X(t) + B(t)\,U(t) \qquad \ldots(1)$$

where $X(t)$ is an n vector, representing the instantaneous state of the system, $u(t)$ is an r-vector ($r \leqslant n$) representing the control input to the system, $A(t)$ is $(n \times n)$ matrix and $B(t)$ is an $(n \times r)$ matrix] has received considerable attention in the literature. The solution of the above equation can be expressed in the following integral forms :

$$X(t) = \phi(t, t_0)\; X(t_0) + \int_{t_0}^{t} \phi(t, s)\, B(s)\, U(s)\, ds \qquad \ldots(2)$$

where $\phi(t, t_0)$ is the fundamental matrix of the system (1), and $x(t_0)$, the initial state of the system at time $t = t_0$. The minimum time control problem, is to find the optimal control $u(t)$ belonging to the admissible set, which will derive the systems from a given initial state $x(t_0)$ at $t = t_0$, to the desired state x_1 in minimum time t i. e. $x(t) = x_1$. Now (2) can be written as $X(t) - \phi(t, t_0) X(t_0) = \int_{t_0}^{t} \phi(t, s) B(s) U(s) ds$. Put $X(t) - \phi(t, t_0) X(t_0) = \xi$.

Expression (2) can be written as $\xi = T_t u$

where

$$T_t u = \int_{t_0}^{t} \phi(t, s) B(s) U(s) ds.$$

Thus without any loss of generality one can consider the problems of finding the optimal u to drive the system from the origin to any point ξ in minimum time t.

The above problem can be considered as a mapping from some space to which u belongs to some other space to which ξ belongs. In the light of the above we can consider following general problem :

Let B_t be a Banach space depending on the parameter t and D be also a Banach space. Let T_t be a bounded linear transformation depending on the parameter mapping B_t on-to D. Let $\xi \in D$. The problem is to find the optimal control $u(.) \in B_t$ to reach ξ from the origin in minimum time t under the constraint $\|u(.)\| \leqslant 1$ where $\|u(.)\|$ denotes the norm of $u(.)$ in B_t.

In many practical problems in engineering and technology, it turns out that one can identify a suitable function space to which the control function belongs, such that constraint turns out precisely to be the constraint on the norm of u in that above space (Ref. 9).

Chaudhuri and Mukherjee (Ref. 2, 3), considered the minimum control for systems which can be represented as a mapping from a Banach space of control functions to another Banach space of the state of the control systems. The Banach space of control functions were either a Reflexive space or a conjugate of some other Banach space.

Problem Statement

In this paper we shall consider the problem where the constraints on the control function are given as follows :

$$|u(\tau)| \leqslant N, \left(\int_{0}^{t} |u(\tau)|^2 d\tau\right)^{1/2} \leqslant M$$

M and N being to positive constraints. The problem is to find the optimal control function u which will drive the origin (initial state) to ξ (desired state) in minimum time t, satisfying the above constraints.

For the sake of completeness, we shall now give certain definitions, theorems and lemmas.

Definitions—The set of all points $\xi \in D$, such that $T_t u = \xi$ for some $u \subseteq U_t \subset B_t$ will be called the Reachable set and will be denoted by $C(t)$, where U_t is the unit ball in B_t, for some given time t.

In following theorems B_t, T_t, D, will mean the same as defined earlier, until they are specifically defined.

Theorem 1—If B_t and D be the conjugate spaces of the normed linear spaces X_t and Y respectively and T_t is the adjoint of some bounded linear transformation S, mapping Y one to one and onto a closed subspace of X_t, then $C(t)$ is closed.

PROOF : By Alaglu's theorem[7,8] the unit ball in X_t^* is weak* compact. Also, both X_t^* and D are equipped with their weak* topologies. Again, as T_t is adjoint to S, T_t will also be onto and remains continuous with respect to weak* topologies of X_t^* and D. Consequently, the unit ball of X_t^* will be mapped onto a weak* compact subset of D. Hence $C(t)$ is weak* closed, and therefore weakly closed and hence norm closed in D.

Note : Let X be a normed linear space and X^* be its conjugate. Hahn-Banach theorem[7,8] assures that, for each $x (\neq \theta) \in X$ there exists an $f \in X^*$ which satisfies the condition $f(x) = \|x\|$ and $\|f\| = 1$. Such an f will be called an *extremal* of x. (X^* denotes the conjugate space of the normed linear space X).

Note : The Reachable set is also convex body, symmetric with respect to the origin of D (Ref. 3).

Theorem 2—Let B_t be the conjugate space of the normed linear space X_t and D is the conjugate of some normed linear space Y. Let $\xi \in \delta C(t)$, where $\delta C(t)$ denotes the boundary of $C(t)$ for some given time t. Then there exists at least one $u_\xi(t) \in U_t \subset B_t$ which will transfer the system from origin to $\xi \in \delta C(t)$ in minimum time t, where T_t is as in Theorem 1.

PROOF : Here, $S : Y \to X_t$, $S^* : X_t^* \to Y^*$ i. e. $S^* : B_t \to D \therefore S^* = T_t$. If $\phi \in D^*$, then $T_t^* \phi \in B_t^*$, $\therefore T_t^* \phi \in B_t^{**}$ where $\overline{T_t^* \phi}$ denotes the extremal of $T_t^* \phi$; where $\|\overline{T_t^* \phi}\| = 1$.

Since, T_t is onto, (ref. 5, 9), hence by the open mapping theorem, the optimal control $u_\xi \in U_t \subset B_t$. Hence there exists at least one $u_\xi(t) = \overline{T_t^* \phi} \in U_t \subset B_t \subset B_t^{**}$. Now if t^* is the minimum time to reach ξ, then $\xi \in \delta C(t^*)$. Let $\phi_\xi \in D^*$ be the supporting hyperplane to $\delta C(t^*)$ at ξ. Let u_ϕ be the optimal control to reach ξ in minimum time t^* – then $u_\phi = \overline{T_t^* \phi}$, $\|u_\phi\| = 1$ (see ref 2, 3). See Burns' *et al.* for determining ϕ_ξ and t^* for a given ξ.

Remark 1 : If D in finite dimensional, then S always exist. We state the following lemmas which can be easily proved.

Lemma 1—Let X be a normed linear space. If $\rho_1(x)$ and $\rho_2(x)$ are the seminorm and norms respectively in X, then, Max $\{\rho_1(x), \rho_2(x)\}$ is a norm in X, where $x \in X$.

Corollary—Evidently Max $\{\rho_1(x), \rho_2(x)\}$, is a norm, where each of $\rho_1(x)$, $\rho_2(x)$ is a norm.

Lemma 2—$$\|u\|_{\infty,N,2,M} = \text{Max}\left\{\operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} \frac{|u(\tau)|}{N}, \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2}\right\}$$

is equivalent to $\|u\|_\infty = \operatorname*{ess\,sup}_{0\leqslant\tau<t} |u(\tau)|$ which is a norm on $L_\infty(0, t)$.

PROOF : We have $$\frac{1}{M}\left\{\int_0^t |u(\tau)|^2 d\tau\right\}^{1/2} \leqslant \frac{1}{M} \operatorname*{ess\,sup}_{0<\tau\leqslant t} |u(\tau)| \cdot \sqrt{t}$$

$$= \frac{N}{M}\left(\operatorname*{ess\,sup}_{0<\tau\leqslant t} \frac{|u(\tau)|}{N}\right)\sqrt{t}.$$

We shall consider two cases, case (i) and case (ii), and two subcases of case (ii).

Case (i)—If $t \leqslant \frac{M^2}{N^2}$, $$\frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2} \leqslant \operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} \frac{|u(\tau)|}{N}$$

$$\therefore \|u\|_{\infty,N,2,M} = \operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} \frac{|u(\tau)|}{N} = \|u\|_{\infty,N}, \text{ for } t \leqslant \frac{M^2}{N^2}. \qquad \ldots(3)$$

Hence $\|u\|_{\infty,N,2,M}$ is equivalent to $\|u\|_\infty$ for $t \leqslant \frac{M^2}{N^2}$.

Case (ii)—$t > \frac{M^2}{N^2}$. There will be two subcases :

$$\text{(a)} \quad \operatorname*{ess\,sup}_{0 \leqslant \tau \leqslant t} \frac{|u(\tau)|}{N} = \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2}$$

on a set of finite measure.

$$\text{(b)} \quad \operatorname*{ess\,sup}_{0 \leqslant \tau \leqslant t} \frac{|u(\tau)|}{N} < \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2} \text{ almost everywhere.}$$

We shall make use of the following notations :

$$\|u\|_{\infty,N} = \operatorname*{ess\,sup}_{0 \leqslant \tau \leqslant t} \frac{|u(\tau)|}{N}, \quad \|u\|_{2,M} = \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2}.$$

Obviously $\|u\|_{\infty,N}$ and $\|u\|_{2,M}$ are norms and they are equivalent to $\|u\|_\infty$ and $\|u\|_2$ respectively.

In *case* (*ii*) (*a*) : $\|u\|_{\infty,N,2,M} = \|u\|_{\infty,N} = \|u\|_{2,M}$

$$\therefore \|u\|_{\infty,N,2,M} \leqslant \|u\|_{\infty,N} \leqslant \|u\|_{\infty,N,2,M} \qquad \ldots(4)$$

In *case* (*ii*) (*b*) : $\|u\|_{\infty,N,2,M} = \|u\|_{2,M} \leqslant \operatorname*{ess\,sup}_{0 \leqslant \tau \leqslant t} \frac{|u(\tau)|}{M} \cdot \sqrt{t}$

or

$$\frac{M}{N\sqrt{t}} \|u\|_{\infty,N,2,M} \leqslant \operatorname*{ess\,sup}_{0 \leqslant \tau \leqslant t} \frac{|u(\tau)|}{N} = \|u\|_{\infty,N} < \|u\|_{2,M}$$

$$= \|u\|_{\infty,N,2,M} \qquad \ldots(5)$$

combining (3), (4) and (5) we obtain

$$\text{Max}\left\{1, \frac{M}{N\sqrt{t}}\right\} \|u\|_{\infty,N,2,M} \leqslant \|u\|_{\infty,N} \leqslant \|u\|_{\infty,N,2,M}. \qquad \ldots(6)$$

But $\|u\|_{\infty,N}$ is obviously equivalent to $\|u\|_\infty$. Hence from (6) $\|u\|_{\infty,N,2,M}$ is equivalent to $\|u\|_\infty$. Hence the proof.

Definition—We define $L_{\infty,N,2,M}$ to be the space of all essentially bounded functions u, equipped with the norm $\|u\|_{\infty,N,2,M}$.

Definition—We define $L_{\infty,N}$ to be the space of all essentially bounded functions u, equipped with the norm $\|u\|_{\infty,N} = \underset{0\leqslant\tau\leqslant t}{\text{ess sup}} \frac{|u(\tau)|}{N}$.

Definition—The space $L_{2,M}$ consists of all square integrable functions u, equipped with the norm $\|u\|_{2,M} = \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2}$.

Note : Evidently $\|u\|_{\infty,N}$ and $\|u\|_{2,M}$, are equivalent to $\|u\|_\infty$ and $\|u\|_2$ respectively and hence the spaces $L_{\infty,N}$ and $L_{2,M}$ are complete with respect to their respective norms $\|u\|_{\infty,N}$ and $\|u\|_{2,M}$.

Consider a system described by (1) where $u(t)$ is a scalar control. Assume that at $t = 0$ the state of the system is given be $x(0)$. It is required to find $u(t)$ which will bring the system from the initial state $x(0)$ to the orign of the state space in the least time under the constraint $|u(\tau)| \leqslant N\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2} \leqslant M$.

The above constraints can be expressed in the following alternative forms :

$$J(u) = \text{Max}\left\{\underset{0\leqslant\tau\leqslant t}{\text{ess sup}} \frac{|u(\tau)|}{N}, \frac{1}{M}\left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2}\right\}.$$

From Lemma 1, it follows that $J(u)$ is a norm in $L_{\infty,N,2,M}$.

Now $L_{\infty,N,2,M}$ can be considered as the conjugate of the space $L_{1,N,2,M}$ i. e. $L_{1,N,2,M} = L_{\infty,N,2,M}$ where * denotes the conjugate of the corresponding spaces. Here $T_t : L_{\infty,N,2,M} \to R^n$ where R^n denotes the n-dimensional Euclidean space. In the finite dimensional case it can be easily shown that $T_t^* = S$ is one to one and onto a closed subspace of $L_{1,N,2,M}$, where $S : R^n \to L_{1,N,2,M}$. By Theorem 1 one can easily verify that the corresponding Reachable set is closed. Also, By Theorem 2, it follows that there exists an optimal control u_ϕ.

The Form of the Optimal Control

The problem is to find u which will maximize $< u, T_t^* \phi >$.

Under the constraint $| u(\tau) | \leqslant N$, $\frac{1}{M}\left(\int_0^t | u(\tau) |^2 d\tau\right)^{1/2} \leqslant 1$.

Case (I)—If $t \leqslant \frac{M^2}{N^2}$ then $\|u\|_{\infty,N,2,M} = \operatorname{ess\,sup}_{0\leqslant\tau\leqslant t} \frac{| u(\tau) |}{N} = 1$

$$\therefore \operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} | u(\tau) | = N.$$

Now, the optimal u must satisfiy the condition[2,3,4,9] $< u, T_t^* \phi > = \|T_t^* \phi\|_{1,N}$ and $\|u\|_{\infty,N} = 1$. So the problem is to find a u, which will maximize

$$< u, T_t^* \phi > = \int_0^t u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau$$

subject to $\operatorname*{ess\,sup}_{0\leqslant\tau\leqslant\tau\leqslant t} | u(\tau) | = N.$

Evidently the optimal $u(t)$ will be given by $u_\phi(\tau) = N \operatorname{sign}\left[T_t^* \phi(\tau) \right], 0\leqslant\tau\leqslant t$

and $< u, T_t^* \phi > = N \int_0^t | \left(T_t^* \phi \right)(\tau) |\, d\tau.$

It can be easily verified that $\|T_t^* \phi\|_{1,N} = N \int_0^t | T_t^* \phi(\tau) |\, d\tau.$

Case (II) (a)—$\|u\|_{\infty,N,2,M} = \operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} \frac{| u(\tau) |}{N} = \frac{1}{M}\left(\int_0^t | u(\tau) |^2 d\tau\right)^{1/2} \leqslant 1.$

Hence $\operatorname*{ess\,sup}_{0\leqslant\tau\leqslant t} | u(\tau) | = N$ and $\int_0^t | u(\tau) |^2 d\tau = M^2$.

Consequently, one has to find that $u(\tau)$ which will maximize $< u, T_t^* \phi > = \int_0^t u(\tau)$

$\left(T_t^* \phi \right)(\tau)\, d\tau$. Let $E = \{t: \mid u(\tau) \mid = N\}$ and $E_c = \{t : \mid u(\tau) \mid < N\}$

$$\therefore \int_0^t u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau = \int_E u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau + \int_{E_c} u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau.$$

Now $\int_E u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau$ will be maximized if $u(\tau) = N \operatorname{sign} \left[\left(T_t^* \phi \right)(\tau) \right]$, $\tau \in E$.

Again $\int_0^t \mid u(\tau) \mid^2 d\tau = M^2$ i. e $\int_E \mid u(\tau) \mid^2 d\tau + \int_{E_c} \mid u(\tau) \mid^2 d\tau = M^2$

or $\int_{E_c} \mid u(\tau) \mid^2 d\tau = M^2 - N^2 m(E)$

where $m(E)$ denotes the measure of the set E. So $\int_{E_c} u(\tau) \left(T_t^* \phi \right)(\tau)\, d\tau$ will be maximized under the constraint (A), if we take $u(\tau) = \alpha \left(T_t^* \phi \right)(\tau)$ where α is a positive constant. Substituting $u = \alpha \left(T_t^* \phi \right)(\tau)$ in (A), we have

$$\alpha^2 \int_{E_c} \left(\mid T_t^* \phi \right)(\tau) \mid^2 d\tau = M^2 - N^2 m(E)$$

where

$$\alpha = \frac{\sqrt{M^2 - N^2 m(E)}}{\sqrt{\int_{E_c} \mid T_t^* \phi(\tau) \mid^2 d\tau}}$$

$$\text{Max} < u, T_t^* \phi > = N \int_E \mid \left(T_t^* \phi \right)(\tau) \mid d\tau + \sqrt{M^2 - N^2 m(E)} \sqrt{\int_{E_c} \mid T_t^* \phi(\tau) \mid^2 d\tau}.$$

It can be easily verified that

$$\| T_t^* \phi \|_{1,N,2,M} = N \int_E | T_t^* \phi(\tau) | d\tau + \sqrt{M^2 - N^2 m(E)} \sqrt{\int_{E_c} | T_t^* \phi(\tau) |^2 d\tau}$$

from the above it follows that

$$u_\phi(\tau) = N \operatorname{Sign}\left[\alpha \left(T_t^* \phi \right)(\tau) \right],$$

$$\tau \in E = \left\{ t : | \alpha \left(T_t^* \phi \right)(\tau) | > N \right\}$$

$$= \alpha \left(T_t^* \phi \right)(\tau), \quad \tau \in E_c = \left\{ t : | \alpha \left(T_t^* \phi(\tau) | \leqslant N \right\}.$$

Case (II) (b)—$\|u\|_{\infty,N,2,M} = \frac{1}{M} \left(\int_0^t | u(\tau) |^2 d\tau \right)^{1/2} = 1$

or,

$$\int_0^t | u(\tau) |^2 d\tau = M^2. \qquad \text{...(B)}$$

Now, the problem becomes, find u which will maximize $\int_0^t u(\tau) \left(T_t^* \phi \right)(\tau) d\tau$ under the constraint (B).

Obviously $u_\phi = \alpha T_t^* \phi$, such that $\alpha^2 \int_0^t | T_t^* \phi(\tau) |^2 d\tau = M^2$

i. e.

$$\alpha = \frac{M}{\sqrt{\int_0^t | T_t^* \phi(\tau) |^2 d\tau}}.$$

Thus

$$u_\phi(\tau) = \alpha\, T_t^*\, \phi(\tau) = \frac{M\, T_t^*\, \phi(\tau)}{\sqrt{\int_0^t |\, T_t^*\, \phi(\tau)\,|^2\, d\tau}}$$

and

$$\int_0^t u(\tau)\left(T_t^*\,\phi\right)(\tau)\, d\tau = \alpha \int_0^t |\, T_t^*\, \phi(\tau)\,|^2\, d\tau$$

$$= M\left(\int_0^t |\, T_t^*\, \phi(\tau)\,|^2\, d\tau\right)^{1/2}$$

$$= \|\, T_t^*\, \phi\|_{1,N,2,M}.$$

Example—In positional control by separately, excited D. C. motor the load is moved to the desired position by supplying armeture current to the D. C. motor.

The differential equation of the system is

$$\frac{dwm}{dt} = -\frac{Bme}{Jme}\, wm(t) + \frac{Ri}{Jme}\, i\,a(t)$$

$$\frac{d\theta m}{dt} = Wm$$

where J_{me} is the effective inertia of the motor; B_{me} the effective viscous frictional coefficient; θ_m the angular position of the motor; R_t the torque constant; and i_a the armeture current and is the control. In such system the amplitude of the current should be limitted. Also it is desirable the power-loss due to dissipiation in the motor should not be permitted to exceed a given bound—which can be expressed as $\{\int_0^t |\, i_a(\tau)\,|^2\, d\tau\}^{1/2} \leqslant M^2$.

Thus the problem is to find the control $i_a(t)$ so that system can be transferred from any position to any other position in minimum time. For a particular controller, $B_{me} = 10^{-3}$ ib ft | sec, $J_{me} = 2.10^{-3}$ ib ft/sec^2 $R_t =$ ib ft/amp.

Hence the system can be written as

$$\frac{d}{dt}\begin{bmatrix} \theta_m \\ w_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -.5 \end{bmatrix}\begin{bmatrix} \theta_m \\ w_m \end{bmatrix} + \begin{bmatrix} 0 \\ 250 \end{bmatrix} i_a. \qquad \ldots(C)$$

The solution of (C) can be written as

$$e^{-At}\begin{bmatrix} \theta_m(t) \\ w_m(t) \end{bmatrix} - \begin{bmatrix} \theta_m(0) \\ w_m(0) \end{bmatrix} = \int_0^t e^{-As} B\, U(s)\, ds \qquad \ldots(D)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 250 \end{bmatrix}.$$

If the system reaches the null state at time t, the equation (D) reduces

$$-\begin{bmatrix} \theta_m(0) \\ w_m(0) \end{bmatrix} = \int_0^t \begin{bmatrix} 250S \\ 250(1 - .5s) \end{bmatrix} i_a\, ds.$$

Putting

$$-\begin{bmatrix} \theta_m(0) \\ w_m(0) \end{bmatrix} = \xi$$

$$\therefore \xi = \int_0^t \begin{bmatrix} 250\ S \\ 250(1 - .5s) \end{bmatrix} i_a\, ds = T_t\, u.$$

$$\therefore T_t^*\, \phi_\xi = 250\begin{bmatrix} t, & 1 - .5t \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 250\,(t\phi_1 + (1 - .5t)\,\phi_2)$$

$$\therefore u_\phi(t) = N \text{ Sign } [\alpha\ (250\, t\phi_1 + 250\,(1 - .5t)\,\phi_2],\ t \in E$$

$$= \alpha\,[250\, t\phi_1 + 250\,(1 - .5t)\,\phi_2].\ \ t \in E_c$$

where

$$\alpha = \frac{\sqrt{M^2 - N^2\, m(E)}}{\sqrt{\int_{E_c} (250)^2\,[t\,\phi_1 + (1 - .5t)\,\phi_2]^2\, dt}}$$

$$E = \{t : \alpha\,[250\, t\,\phi_1 + 250\,(1 - .5t)\,\phi_2] > N\}$$

$$E_c = \{t : \alpha\,[250\, t\,\phi_1 + 250\,(1 - .5t)\,\phi_2] \leqslant N\}.$$

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