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ON FIXED POINT RESULTS

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In this paper some fixed point theorems for a family of self maps are obtained to generalise a recent result of Maiti and Ghosh and also some results of Nirmala Bajaj are modified.

In this paper, we prove two unique common fixed point theorems for a family of self maps. One of which improve the result of Maiti and Ghosh[4] and the other generalize the results of Iseki[3] and Chattopadyay[1]. We observe that two results of Bajaj[5] are not valid and hence we suggest a suitable modification. Finally, we observe that the decreasing property of the sequence in Theorem 7 of Rao and Rao[6] is not necessary.

Theorem 1—Let X be a nonempty compact Hausdorff space, $g : X \to X$ be such that for some positive integer p, g^r is continuous and $\mathcal{F}$ be a nonempty family of self maps on X such that $g^p S = S g^p$ for every $S \in \mathcal{F}$. Let $F : X \times X \to R_+$ be continuous and $F(x, x) = 0$ for all $x \in X$. Further assume that for some positive integer r,

$$F(g^r x, g^r y) < \delta [\mathcal{G}(x) \cup \mathcal{G}(y)] \text{ for } x \neq y, \qquad \ldots(1)$$

where $\mathcal{G}(x) = \{h x \mid h \in \tau\}$, τ being the semigroup of self maps on X generated by $\mathcal{F} \cup \{g, I\}$ (I, the identity map on X).

Then the family $\mathcal{F} \cup \{g\}$ has a unique common fixed point $z \in X$ and z is the only fixed point of g.

PROOF : Define $H = \bigcap_{n=1}^{\infty} g^{np} X$.

Then, as in Theorem 4 of Rao and Rao[6], H is a nonempty compact subset of X, $gH = H$ and $SH \subseteq H$ for all $S \in \mathcal{F}$.

There exist $z_1, z_2, x_1, x_2 \in H$ such that

$$\delta(H) = F(z_1, z_2) = \sup \{F(x, y) \mid x, y \in H\}$$

and

$$z_1 = g^r x_1, z_2 = g^r x_2.$$

Suppose $x_1 \neq x_2$. Then by (1), we have

$$\delta(H) = F(g^r x_1, g^r x_2) < \delta[\mathcal{O}(x_1) \cup \mathcal{O}(x_2)] \leqslant \delta(H)$$

a contradiction.

Hence $H = \{z\}$ say.

Thus z is a common fixed point of the family $\mathcal{F} \cup \{g\}$. Since every fixed point of g is a point of $H = \{z\}$, it follows that z is the only fixed point of g and it is the unique common fixed point of $\mathcal{F} \cup \{g\}$.

Theorem 2— Let (X, d) be a complete metric space, g be a continuous and densifying self map on X, $\mathcal{F}$ be a nonempty finite family of commuting, continuous and densifying selfmaps on X such that $gS = Sg$ for every $S \in \mathcal{F}$ and satisfying (1) with F as in Theorem 1. Further assume that for some $x_0 \in X$, $\mathcal{O}(x_0)$ is bounded. Then we have the conclusion of Theorem 1.

PROOF : Write $A = \mathcal{O}(x_0)$.

Since every two members of $\mathcal{F} \cup \{g\}$ are commuting, we have $SA \subseteq A$ for every $S \in \mathcal{F} \cup \{g\}$ and $A = \{x_0\} \cup gA \cup [\bigcup_{S \in \mathcal{F}} SA]$.

Since every $S \in \mathcal{F} \cup \{g\}$ is densifying and (X, d) is complete, it follows that $\bar{A}$ is compact.

Define $H = \bigcap_{n=1}^{\infty} g^n \bar{A}$.

Then the rest of the proof follows as in Theorem 1.

Remark 1 : By taking $\mathcal{F} = \{f\}$ in Theorem 1, we get the result which improves the result of Maiti and Ghosh[4].

Remark 2 : By taking $\mathcal{F} = \{I\}$ in Theorem 2, we get the result which generalize the results of Iseki[3] and Chattopadyay[1].

Recently Bajaj[5] proved the following :

Theorem 3 (Theorem 2 of Bajaj[5])— Let S and T be continuous self maps on a compact metric space (X, d) satisfying

$$d(Sx, Ty) < \alpha \frac{d(x, Sx)\, d(x, Ty) + [d(x, y)]^2 + d(x, Sx)\, d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}$$

for $x \neq y$ and $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$, where $0 < \alpha < 1$. Then S and T have a unique common fixed point. Further if $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies $d(Sx, Ty) = 0$, then S and T have a unique common fixed point.

Unfortunately, it is not valid in view of the following

Example 4—Let $X = \{0, 1\}$ with the usual metric d,

$$S0 = 0,\ S1 = 1,\ T0 = 1,\ T1 = 0.$$

Now we modify Theorem 3 as follows:

Theorem 5—Let X be a pseudo compact Hausdorff space, $F: X \times X \to R_+$ be continuous symmetric, $F(x, x) = 0$ for all $x \in X$ and $F(x, y) \neq 0$ for $x \neq y$. Let $S, T: X \to X$ be such that

either

$$S \text{ or } T \text{ is continuous} \qquad \ldots(2)$$

and

$$F(Sx, Ty) < \frac{F(x, Sx)\, F(x, Ty) + [F(x, y)]^2 + F(x, Sx)\, F(x, y)}{F(x, Sx) + F(x, y) + F(x, Ty)} \qquad \ldots(3)$$

for $x \neq y$, $Sx \neq Ty$ and $x \neq Sx$.

Then either S or T has a fixed point.

PROOF : Assume that S is continuous. Define $\phi: X \to R_+$ by $\phi(x) = F(x, Sx)$ for $x \in X$.

Then ϕ is continuous on X and hence attains its bounds.

There exists $w \in X$ such that $\phi(w) = \inf \{\phi(x) \mid x \in X\}$.

Suppose neither S nor T has a fixed point.

Then by (3), we have

$$\phi(TSw) = F(STSw, TSw) < F(Sw, TSw) < F(w, Sw) = \phi(w)$$

a contradiction.

Hence either S or T has a fixed point.

Remark : In Theorem 5, if we assume the condition (3) for $F(x, Sx) + F(x, y) + F(x, Ty) \neq 0$ then S and T have a unique common fixed point.

Theorem 1 of Bajaj[5] is also not valid in view of Example 4. Under the first part hypothesis of Theorem 1 of Bajaj[5], either S or T has a fixed point and if they have a common fixed point, it is unique. The second part hypothesis is not necessary.

Theorem 3 of Bajaj[5] can be improved as follows

Theorem 6—Let S and T be self maps on a Hausdorff space,

$F: X \times X \to R_+$ be such that $F(x, x) = 0$ for all $x \in X$ and $F(x, y) \neq 0$ for $x \neq y$. Also assume

$$F(Sx, Ty) \leqslant \frac{F(x, Sx)\ F(x, Ty)}{F(x, Sx) + F(x, y) + F(x, Ty)} \text{ for } F(x, Sx)$$

$$+ F(x, y) + F(x, Ty) \neq 0.$$

Then S and T have a unique common fixed point $z \in X$ such that $z = Tx$ for all

$$x \in X.$$

Finally we conclude that the decreasing property of sequence in Theorem 7 of Rao and Rao[6] is not necessary in view of the proof of Theorem 1 of Chatterjee[2].

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