ON FIXED POINT RESULTS

K. P. R RAO

Department of Mathematics, D. A. R. College, Nuzvid 521201

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In this paper some fixed point theorems for a family of self maps are obtained to generalize a recent result of Maiti and Ghosh and also some results of Nirmala Bajaj are modified.

In this paper, we prove two unique common fixed point theorems for a family of self maps. One of which improve the result of Maiti and Ghosh and the other generalize the results of Iseki and Chattopadhyay. We observe that two results of Bajaj are not valid and hence we suggest a suitable modification. Finally, we observe that the decreasing property of the sequence in Theorem 7 of Rao and Rao is not necessary.

Theorem 1—Let \( X \) be a nonempty compact Hausdorff space, \( g : X \to X \) be such that for some positive integer \( p \), \( g^p \) is continuous and \( \mathcal{F} \) be a nonempty family of self maps on \( X \) such that \( g^pS = Sg^p \) for every \( S \in \mathcal{F} \). Let \( F : X \times X \to \mathbb{R}_+ \) be continuous and \( F(x, x) = 0 \) for all \( x \in X \). Further assume that for some positive integer \( r \),

\[
F(g^r x, g^r y) < \delta [\mathcal{G}(x) \cup \mathcal{G}(y)] \text{ for } x \neq y,
\]

where \( \mathcal{G}(x) = \{ h x | h \in \tau \} \), \( \tau \) being the semigroup of self maps on \( X \) generated by \( \mathcal{F} \cup \{ g, I \} \) (\( I \), the identity map on \( X \)).

Then the family \( \mathcal{F} \cup \{ g \} \) has a unique common fixed point \( z \in X \) and \( z \) is the only fixed point of \( g \).

Proof: Define \( H = \bigcap_{n=1}^{\infty} g^n p X \).

Then, as in Theorem 4 of Rao and Rao, \( H \) is a nonempty compact subset of \( X \), \( gH = H \) and \( SH \subseteq H \) for all \( S \in \mathcal{F} \).

There exist \( z_1, z_2, x_1, x_2 \in H \) such that
\[ \delta(H) = F(z_1, z_2) = \sup \{F(x, y) \mid x, y \in H\} \]

and

\[ z_1 = g' x_1, z_2 = g' x_2. \]

Suppose \( x_1 \neq x_2 \). Then by (1), we have

\[ \delta(H) = F(g' x_1, g' x_2) < \delta[\mathcal{G}(x_1) \cup \mathcal{G}(x_2)] \leq \delta(H) \]

a contradiction.

Hence \( H = \{z\} \) say.

Thus \( z \) is a common fixed point of the family \( \mathcal{F} \cup \{g\} \). Since every fixed point of \( g \) is a point of \( H = \{z\} \), it follows that \( z \) is the only fixed point of \( g \) and it is the unique common fixed point of \( \mathcal{F} \cup \{g\} \).

**Theorem 2**—Let \((X, d)\) be a complete metric space, \( g \) be a continuous and densifying self map on \( X \), \( \mathcal{F} \) be a nonempty finite family of commuting, continuous and densifying selfmaps on \( X \) such that \( gS = Sg \) for every \( S \in \mathcal{F} \) and satisfying (1) with \( F \) as in Theorem 1. Further assume that for some \( x_0 \in X \), \( \mathcal{G}(x_0) \) is bounded. Then we have the conclusion of Theorem 1.

**Proof**: Write \( A = \mathcal{G}(x_0) \).

Since every two members of \( \mathcal{F} \cup \{g\} \) are commuting, we have \( SA \subseteq A \) for every \( S \in \mathcal{F} \cup \{g\} \) and \( A = \{x_0\} \cup gA \cup \bigcup_{S \in \mathcal{F}} SA \).

Since every \( S \in \mathcal{F} \cup \{g\} \) is densifying and \((X, d)\) is complete, it follows that \( \bar{A} \) is compact.

Define \( H = \bigcap_{n=1}^{\infty} g^n \bar{A} \).

Then the rest of the proof follows as in Theorem 1.

**Remark 1**: By taking \( \mathcal{F} = \{f\} \) in Theorem 1, we get the result which improves the result of Maiti and Ghosh.

**Remark 2**: By taking \( \mathcal{F} = \{I\} \) in Theorem 2, we get the result which generalizes the results of Iseki and Chattopadhyay.

Recently Bajaj proved the following:
Theorem 3 (Theorem 2 of Bajaj)—Let $S$ and $T$ be continuous self maps on a compact metric space $(X, d)$ satisfying

$$
d(Sx, Ty) < \alpha \frac{d(x, Sx) d(x, Ty) + [d(x, y)]^2 + d(x, Sx) d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}
$$

for $x \neq y$ and $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$, where $0 < \alpha < 1$. Then $S$ and $T$ have a unique common fixed point. Further if $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ implies $d(Sx, Ty) = 0$, then $S$ and $T$ have a unique common fixed point.

Unfortunately, it is not valid in view of the following

Example 4—Let $X = \{0, 1\}$ with the usual metric $d$,

$$S0 = 0, S1 = 1, T0 = 1, T1 = 0.$$  

Now we modify Theorem 3 as follows:

Theorem 5—Let $X$ be a pseudo compact Hausdorff space, $F: X \times X \rightarrow R_+$ be continuous symmetric, $F(x, x) = 0$ for all $x \in X$ and $F(x, y) \neq 0$ for $x \neq y$. Let $S, T: X \rightarrow X$ be such that

either

$S$ or $T$ is continuous

...(2)

and

$$F(Sx, Ty) < \frac{F(x, Sx) F(x, Ty) + [F(x, y)]^2 + F(x, Sx) F(x, y)}{F(x, Sx) + F(x, y) + F(x, Ty)}$$

...(3)

for $x \neq y, Sx \neq Ty$ and $x \neq Sx$.

Then either $S$ or $T$ has a fixed point.

Proof: Assume that $S$ is continuous. Define $\phi: X \rightarrow R_+$ by $\phi(x) = F(x, Sx)$ for $x \in X$.

Then $\phi$ is continuous on $X$ and hence attains its bounds.

There exists $w \in X$ such that $\phi(w) = \inf \{\phi(x) | x \in X\}$.

Suppose neither $S$ nor $T$ has a fixed point.

Then by (3), we have

$$\phi(TSw) = F(STSw, TSw) < F(Sw, TSw) < F(w, Sw) = \phi(w)$$
a contradiction.

Hence either $S$ or $T$ has a fixed point.

Remark: In Theorem 5, if we assume the condition (3) for $F(x, Sx) + F(x, y) + F(x, Ty) \neq 0$ then $S$ and $T$ have a unique common fixed point.

Theorem 1 of Bajaj$^5$ is also not valid in view of Example 4. Under the first part hypothesis of Theorem 1 of Bajaj$^5$, either $S$ or $T$ has a fixed point and if they have a common fixed point, it is unique. The second part hypothesis is not necessary.

Theorem 3 of Bajaj$^5$ can be improved as follows

Theorem 6—Let $S$ and $T$ be self maps on a Hausdorff space, $F: X \times X \to \mathbb{R}_+$ be such that $F(x, x) = 0$ for all $x \in X$ and $F(x, y) \neq 0$ for $x \neq y$. Also assume

$$F(Sx, Ty) \leq \frac{F(x, Sx) F(x, Ty)}{F(x, Sx) + F(x, y) + F(x, Ty)}$$

for $F(x, Sx)$

$$+ F(x, y) + F(x, Ty) \neq 0.$$

Then $S$ and $T$ have a unique common fixed point $z \in X$ such that $z = Tx$ for all $x \in X$.

Finally we conclude that the decreasing property of sequence in Theorem 7 of Rao and Rao$^6$ is not necessary in view of the proof of Theorem 1 of Chatterjee$^5$.

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References