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A COMMON FIXED POINT THEOREM FOR THE COMMUTING MAPPINGS

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The main result of this paper is one fixed point theorem for the family of mappings in probabilistic metric spaces. Using this theorem a new proof of one theorem in metric spaces and one fixed point theorem of family of random operators are given.

1. INTRODUCTION

The development of the fixed point theory in probabilistic metric spaces is of the fundamental importance in probabilistic functional analysis. For detailed discussions of probabilistic metric space and their applications we refer to Schweizer and Sklar[7]. In section 2 a few definitions, theorems and conventions will be made to fill in background for the reader. Section 3 is devoted to the main result of the paper, namely, the proof of one common fixed point theorem for the mapping f and the family of mappings $\{g_i\}_{i\in N}$. In section 4 is given a new proof of the theorem from Chang[10] and some application of the first theorem to the problem of the common fixed point of the family of random operators.

2. PRELIMINARIES

Let R denote the reals and $R^+ = \{x \in R : x > 0\}$. A mapping $F : R \to R^+$ is called a distribution function if it is nondecreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by Δ the set of all distribution functions.

Definition 1—A probabilistic metric space (*PM*-space) is an ordered pair $(X, \mathcal{F})$, where X is an abstract set of elements and $\mathcal{F}$ is a mapping of $X \times X$ into Δ. We shall denote the distribution function $\mathcal{F}(x, y)$ by $F_{x,y}$ and $F_{x,y}(\epsilon)$ will represent the value of $F_{x,y}$ at $\epsilon \in R^+$. The functions $F_{x,y}$, $x, y \in X$, are assumed to satisfy the following conditions :

1. $F_{x,y}(\epsilon) = 1$ for all $\epsilon > 0$, if and only if $x = y$.
2. $F_{x,y}(0) = 0$.

3. $F_{x,y} = F_{y,x}$.

4. If $F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\epsilon + \delta) = 1$.

Throughout this paper H will always denote the specific distribution function defined by

$$H(\epsilon) = \begin{cases} 0, & \epsilon < 0 \\ 1, & \epsilon > 0. \end{cases}$$

Definition 2—A commutative, associative and nondecreasing mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is a T-norm iff $t(a, 1) = a$ for all $a \in [0, 1]$ and $t(0, 0) = 0$.

Definition 3—A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space, t is a T-norm and the generalized triangle inequality

$$F_{x,y}(\epsilon + \delta) \geq t(F_{x,z}(\epsilon), F_{z,y}(\delta))$$

holds for all $x, y, z \in X$ and all $\epsilon, \delta \in R^+$.

The concept of a neighbourhood in a PM-space was introduced by Schweizer and Sklar[7]. If $x \in X$, $\epsilon > 0$ and $\lambda \in (0, 1)$, then an (ϵ, λ)-neighbourhood of x, noted by $U_x(\epsilon, \lambda)$ is defined by

$$U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

If t is continuous then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family $\{U_x(\epsilon, \lambda) : x \in X, \epsilon > 0, \lambda \in (0, 1)\}$ of neighbourhoods.

Definition 4—A set $M \subset (X, \mathcal{F}, t)$ is called probabilistic bounded iff

$$\sup_{\epsilon > 0} \inf_{x,y \in M} F_{x,y}(\epsilon) = 1.$$

3. Common Fixed Point Theorem

Throughout this section we assume that $(X, \mathcal{F}, t)$ is a complete Menger space, t is continuous T-norm which is not weaker then $t_1 = \max\{\text{sum-1}, 0\}$, i. e. $t(a, b) \geq t_1(a, b)$ for all $a, b \in [0, 1]$. The following result is due to Schweizer and Sklar[7].

Lemma 1—Let $(X, \mathcal{F}, t)$ and t be the same as above. Suppose that $x_n \to x$ and $y_n \to y$ and that $F_{x,y}$ is continuous at $\epsilon_0 \in R^+$. Then

$$\lim_{n \to \infty} F_{x_n, y_n}(\epsilon_0) = F_{x,y}(\epsilon_0).$$

Let the function $\phi(\epsilon)$ satisfies the following conditions (ϕ). $(\phi) : \phi : R^+ \to R^+$ is strictly increasing, and $\lim_{n \to \infty} \phi^n(\epsilon) = \infty$ for all $\epsilon > 0$, where $\phi^n(\epsilon)$ denotes the nth iteration of $\phi(\epsilon)$.

Theorem 1—Let $(X, \mathscr{F}, t)$ and ϕ be the same as above. Let $f: X \to X$ be the mapping such that for some $m \in N$, f^m is continuous, let $\{g_n\}_{n\in N}$ be the family of self-mappings of $(X, \mathscr{F}, t)$ such that $g_n f^{m-1}(X) \subset f^m(X)$ and $fg_n = g_n f$ for all $n \in N$ and let there exists a sequence $\{m_n\}_{n\in N}$, $m_n \in N$ such that for all $n, k \in N$ and all $x, y \in f^{m-1}(X)$

$$F_{g_n^{m_n} x,\, g_k^{m_k} y}(\epsilon) \geqslant \min_{u,v \in \left\{fx, fy, g_n^{m_n} x, g_k^{m_k} y\right\}} F_{u,v}(\phi(\epsilon)). \qquad \ldots(1)$$

If there exists $x_j \in f^{m-1}(X)$ such that the sequence $\{y_n\}_{n\in N}$ formed by

$$y_n = g_n^{m_n} x_n = f x_{n+1}$$

is probabilistic bounded, then there exists a unique common fixed point for the family $\{g_n\}_{n\in N}$ and f.

PROOF: First, we shall show that the sequence $\{y_n\}_{n\in N}$ is a Cauchy sequence.

We have

$$F_{y_n, y_k}(\epsilon) = F_{g_n^{m_n} x_n,\, g_k^{m_k} x_k}(\epsilon)$$

$$\geqslant \min_{u,v \in \left\{fx_n fx_k, g_n^{m_n} x_n, g_k^{m_k} x_k\right\}} F_{u,v}(\phi(\epsilon))$$

$$= \min_{u,v \in \{y_{n-1}, y_{k-1}, y_n, y_k\}} F_{u,v}(\phi(\epsilon))$$

for all $\epsilon > 0$ and all $n, k \in N$. The last relation implies that

$$\inf_{i\leqslant n,k} F_{y_n, y_k}(\epsilon) \geqslant \inf_{i-1\leqslant n,k} F_{y_n, y_k}(\phi(\epsilon)) \geqslant \ldots$$

$$\geqslant \inf_{1\leqslant n,k} F_{y_n, y_k}(\phi^{i-1}(\epsilon)).$$

The sequence $\{y_n\}_{n\in N}$ is probabilistic bounded and $\phi^{i-1}(\epsilon) \to \infty$ for $i \to \infty$ and so letting $i \to \infty$ we have that

$$\lim_{i\to\infty} \inf_{i\leqslant n,k} F_{y_n, y_k}(\epsilon) \geqslant \lim_{i\to\infty} \sup_{t<\phi^{i-1}(\epsilon)} \inf_{0\leqslant n'k} F_{y_n, y_k}(t) = H(\epsilon)$$

for all $\epsilon > 0$, which means that $\lim_{n\to\infty} F_{y_n, y_{n+p}}(\epsilon) = H(\epsilon)$ i. e. the sequence $\{y_n\}_{n\in N}$ is a Cauchy sequence.

By the completeness of X we can suppose that $\lim_{n\to\infty} y_n = z = \lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} g_n^{m_n} x_n$. Since f^m is a continuous mapping and f commutes with g_n, we have that

$$\lim_{n\to\infty} f^m fx_{n+1} = \lim_{n\to\infty} f^m g_n^{m_n} x_n = \lim_{n\to\infty} f^m y_n$$

$$= \lim_{n\to\infty} g_n^{m_n} f^{m-1} y_{n-1} = f^m z.$$

To prove that $g_k^{m_k} f^{m-1}z = f^m z$ for all $k \in N$ we shall show that the sequence $\{f^m y_n\}_{n\in N}$ coverges to $g_k^{m_k} f^{m-1}z$.

$$F_{f^m y_n,\ g_k^{m_k} f^{m-1}z}(\epsilon) = F_{g_n^{m_n} f^{m-1} y_{n-1},\ g_k^{m_k} f^{m-1}z}(\epsilon)$$

$$\geqslant \min_{u,\, v \in \left\{ f^m y_{n-1}, f^m z,\ g_n^{m_n} f^{m-1} y_{n-1},\ g_k^{m_k} f^{m-1}z \right\}} F_{u,v}(\phi(\epsilon))$$

$$= \min_{u,\, v \in \left\{ f^m y_{n-1},\ f^m z,\ f^m y_n,\ g_k^{m_k} f^{m-1}z \right\}} F_{u,v}(\phi(\epsilon)).$$

Let R_0 be the set of all points of discontinuity of the distribution function

$$F_{f^m z,\ g_k^{m_k} f^{m-1}z}(\epsilon).$$

Since ϕ^i is strictly increasing, it follows that $R_i = \phi^{-i}(R_0)$ is the set of all discontinuity points of the distribution function $F_{f^m z,\ g_k^{m_k} f^{m-1}z}(\phi^i(\epsilon))$,

$i = 1, 2, 3, \ldots$, and all the sets R_i, $i = 0, 1, 2, \ldots$,

are countable so the set $\bigcup_{i=1}^{\infty} R_i$ is also countable set. If we suppose that the point ϵ is the point of continuity (i. e. $\epsilon \in R^+ \setminus \bigcup_{i=1}^{\infty} R_i$) of the distribution functions

$F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\phi^i(\epsilon))$ for all $i = 0, 1, 2, \ldots$ then we have

$$F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\epsilon) = \lim_{n\to\infty} F_{f^m, y_n,\ g_k^{m_k} f^{m-1} z}(\epsilon)$$

$$\geqslant \lim_{n\to\infty} \min_{u,\, v \in \left\{ f^m y_{n-1},\ f^m z,\ f^m y_n,\ g_k^{m_k} f^{m-1} z \right\}} F_{u,v}(\phi(\epsilon))$$

$$= F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\phi(\epsilon)) = \lim_{n\to\infty} F_{f y_n,\ g_k^{m_k} f^{n-1} z}(\phi(\epsilon))$$

$$\geqslant F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\phi^2(\epsilon)) \geqslant \ldots \geqslant F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\phi^p(\epsilon) = H(\epsilon)$$

for all $\epsilon > 0$ when $p \to \infty$.

If the point ϵ is discontinuity point of the mapping $F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\phi^i(\epsilon))$ for some $i = 0, 1, 2, \ldots$ then there exists the numbers $\epsilon_1, \epsilon_2 \in R^+ \setminus \bigcup_{i=0}^{\infty} R_i$ such that $\epsilon_1 < \epsilon < \epsilon_2$. The distribution is nondecreasing function and so we have

$$1 = H(\epsilon_1) = F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\epsilon_1) \leqslant F_{f^m z,\ g_k^{m_k} f^{m-1} z}(\epsilon)$$

$$\leqslant F_{f^m z, g_k^{m_k} f^{m-1} z}(\epsilon_2) = H(\epsilon_2) = 1.$$

Now, we have that for all $\epsilon > 0$ and all $k \in N$

$$\Gamma_{f^m z,\ g_k^{m_k} f^{m-1} z}(\epsilon) = H(\epsilon)$$

which means that $g_k^{m_k} f^{m-1} z = f^m z$ for all $k \in N$.

Next relation proves that $f^m z$ is a common fixed point of mappings $\left\{ g_k^{m_k} \right\}$ $k \in N$.

$$F_{g_k^{m_k} f^m z,\, f^m z}(\epsilon) = F_{g_k^{m_k} f^m z,\, g_n^{m} f^{m-1} z}(\epsilon)$$

$$\geqslant \min_{u,\, v \in \left\{ f^{m+1} z,\, f^m z,\, g_k^{m_k} f^m z,\, f^m z \right\}} F_{u,v}(\phi\, \epsilon))$$

$$= \min_{u,\, v \in \left\{ g_k^{m_k} f^m z,\, f^m z,\, g_k^{m_k} f^m z,\, f^m z \right\}} F_{u,v}(\phi(\epsilon))$$

$$= F_{g_k^{m_k} f^m z,\, f^m z}(\phi(\epsilon)) \geqslant \ldots \geqslant F_{g_k^{m_k} f^m z,\, f^m z}(\phi^p(\epsilon)) = H(\epsilon)$$

for all $\epsilon > 0$, all $k \in N$ and $p \to \infty$.

Since

$$f f^m z = f\, g_k^{m_k} f^{m-1} z = g_k^{m_k} f^m z$$

it follows that $f^m z$ is the fixed point of the mapping f.

To establish that $f^m z$ is a common fixed point for the family of mappings $\{g_k\}_{k \in N}$ we proceed as follows. Since

$$g_k f^m z = g_k f^{m+1} z = g_k f\, g_k^{m_k} f^{m-1} z = g_k^{m_k} g_k f^m z$$

we have that

$$F_{g_k f^m z,\, f^m z}(\epsilon) = F_{g_k^{m_k} g_k f^m z,\, g_k^{m_k} f^m z}(\epsilon)$$

$$\geqslant \min_{u,\, v,\, \in \{g_k f^{m+1} z,\, f^{m+1} z,\, g_k f^m z,\, f^m z\}} F_{u,v}(\phi(\epsilon))$$

$$= F_{g_k f^m z,\, f^m z}(\phi(\epsilon)) \geqslant \ldots \geqslant F_{g_k f^m z,\, f^m z}(\phi^p(\epsilon)) = H(\epsilon)$$

for all $\epsilon > 0$, all $k \in N$ and $p \to \infty$.

In order to prove that $f^m z$ is unique common fixed point of the mappings f and $\{g_k\}_{k \in N}$ we propose that there exists another common fixed point $w \in f^{m-1}(X)$. Then we get

$$F_{w,\, f^m z}(\epsilon) = F_{g_k^{m_k} w,\, g_n^{m_n} f^m z}(\epsilon)$$

$$\geqslant \min_{u,\, v \in \left\{ fw,\, f^{m+1}z,\, g_k^{m_k} w,\, g_n^{m_n} f^m z \right\}} F_{u,v}(\phi(\epsilon))$$

$$= F_{w,\, f^m z}(\phi(\epsilon)) \geqslant \ldots \geqslant F_{w,\, f^m z}(\phi^p(\epsilon)) = H(\epsilon)$$

for all $\epsilon > 0$ and $p \to \infty$, and this means that $w = f^m z$.

This completes the proof of Theorem 1.

4. Common Fixed Point in Metric and E-Spaces

If (M, d) is a metric space then the metric d induces a mapping $\mathcal{F} : M \times M \to \Delta$, where $\mathcal{F}(x, y)$ $(x, y \in M)$ is defined by $\mathcal{F}(x, y)(\epsilon) = F_{x,y}(\epsilon) = H(\epsilon - d(x,y))$, $\epsilon \in R$. Further, if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then $(M, \mathcal{F}, t)$ is a Menger space. It is complete if the metric d is complete. The space $(M, \mathcal{F}, \min)$ so obtained will be called induced Menger space. Using some results stated in preceding section we shall study the existence and uniqueness of common fixed point in metric spaces. The next theorem is a generalization of some known results[4,6,11].

Let the function $\psi(\epsilon)$ satisfies the following conditions (ψ):

(ψ) : $\psi : R^+ \to R^+$ is strictly increasing, $\lim_{\epsilon \to \infty} (\epsilon - \psi(\epsilon)) = \infty$,

$\lim_{n \to \infty} \psi^{-n}(\epsilon) = \infty$ for all $\epsilon > 0$.

It is easy to prove that the mapping ψ has the property that $\psi(t) < t$, i. e. that $\phi^{-1}(t) > t$.

Theorem 2—Let (M, d) be a complete metric space and ψ be the same as above. Let $f : M \to M$ be the mapping such that for some $m \in N$, f^m is continuous, let $\{g_n\}_{n \in N}$ be the family of self-mappings of (M, d) such that $g_n f^{m-1}(M) \subset f^m(M)$ and $f g_n = g_n f$ for all $n \in N$ and let there exists a sequence $\{m_n\}_{n \in N}$, $m_n \in N$, such that for all $n, k \in N$, all $x, y \in f^{m-1}(M)$

$$d\left(g_n^{m_n} x,\, g_k^{m_k} y \right) \leqslant \psi \left(\max_{u, v \in \left\{ fx,\, fy,\, g_n^{m_n} x,\, g_k^{m_k} y \right\}} d(u, v) \right). \qquad \ldots(2)$$

Then there exists a unique common fixed point for the family $\{g_n\}_{n\in N}$ and f and all sequence $\{f y_n\}_{n\in N}$ and $\{g_i y_n\}_{n\in N}$ (for all $i\in N$) formed by

$$y_n = g_n^{m_n} x_n = f x_{n+1},\ x_1 \text{ is any element of } f^{m-1}(M)$$

converges to this fixed point.

PROOF: Let x_1 be any element of M. We can form the sequence $\{y_n\}n\in N$ by

$$y_n = g_n^{m_n} x_n = f x_{n+1}.$$

We shall show that every sequence $\{y_n\}_{n\in N}$ is bounded, i. e. that $\{y_n\}_{n\in N}$ is probabilistic bounded in induced Menger space $(M, \mathscr{F}, \min)$.

Let us denote by $D(A)$ the diameter of a set $A \subset M$ $(D(A) = \sup_{x,y\in A} d(x, y))$ and by $\{y_n\}_{n=i}^{j}$ the set $\{y_i, y_{i+1}, \ldots, y_j\}$.

In other to prove that $D(\{y_n\}_{n\in N}) < \infty$ for any $x_i \in M$, we proceed as follows.

Let $y_i, y_j \in \{y_n\}_{n\in N}$ and $i < j$. Then

$$d(y_i, y_j) = d\left(g_i^{m_i} x_i, g_j^{m_j} x_j\right) \leqslant \psi \left(\max_{x,y\in\left\{fx_i, fx_j, g_i^{m_i} x_i, g_j^{m_j} x\right\}} d(x, y)\right)$$

$$= \psi \left(\max_{x,\, y \in \{y_{i-1}, y_{j-1}, y_i, y_j\}} d(x, y)\right) \leqslant \psi\left(D\{y_n\}_{n-i-1}^{j}\right)$$

$$< D\left(\{y\}_{n-i-1}^{j}\right) = d(y_{i-1}, y_k) \text{ where } i - 1 < k \leqslant j.$$

Further, we have

$$D\left(\{y_n\}_{n-1}^{r}\right) = d(y_1, y_k) \leqslant d(y_1, y_2) + d(y_2, y_k) \leqslant d(y_1, y_2)$$

$$+ \psi D\left(\{y_n\}_{n-1}^{k}\right)$$

$$\leqslant d(y_1 y_2) + \psi\left(D\{y\}_{n-1}^{r}\right)$$

which implies next inequality

$$D\left(\{y_n\}_{n=1}^{p}\right) - \psi\left(D\{y_n\}_{n=1}^{p}\right)\Big) \leqslant d(y_1, y_2).$$

Letting $p \to \infty$ and noting the property (ψ) of the mapping ψ we see that $D(\{y_n\}_{n\in N})$ can not be ∞, what means that $D(\{y_n\}_{n\in N}) < \infty$. It is obvious that the mapping $\psi^{-1} = \phi$ has property (ϕ).

To establish that the condition (2) implies the condition (1) in the induced Menger space $(M, \mathcal{F}, \min)$ we prove next implication (for any $A \subset M$):

$$d(a, b) \leqslant \psi\left(\max_{x,y\in A} d(x, y)\right) \Rightarrow F_{a,b}(\epsilon) \geqslant \min_{x,y\in A} F_{x,y}(\psi^{-1}(\epsilon)).$$

$$d(a, b) \leqslant \psi\left(\max_{x,y\in A} d(x, y)\right) \Rightarrow \epsilon - d(a, b) > \epsilon - \psi\left(\max_{x,y\in A} d(x, y)\right)$$

$$\Rightarrow H(\epsilon - d(a, b)) \geqslant H\left(\epsilon - \psi\left(\max_{x,y\in A} d(x, y)\right)\right)$$

$$\Rightarrow H(\epsilon - d(a, b)) \geqslant \min_{x,y,\in A} H(\psi^{-1}(\epsilon) - d(x, y)))$$

$$\Rightarrow F_{a,b}(\epsilon) > \min_{x,y\in A} F_{x,y}(\psi^{-1}(\epsilon)).$$

This completes the proof of Theorem 2.

Sherwood[8] introduced a class of probabailistic metric spaces $(X, \mathcal{F})$ for which X is the set of the functions (random variables) from a probability space (Ω, K, P) into a metric space (M, d) subject to the restriction that, for every $x, y \in X$ and every $\epsilon > 0$ the set A of element $w \in \Omega$ such that $d(x(w), y(w)) < x$ belongs to K, and for which $\mathcal{F}(x, y, w)$ is defined to be $P(A)$. We call these spaces and every E-spaces E-space is a Menger space under the T-norm t_m ($t_m(a, b) = \max\{a + b - 1, 0\}$). This space is a Hausdorff space in the topology $\mathcal{F}$ induced by the family of neighbourhoods defined above. If (M, d) is a separable complete metrics pace and (Ω, K, P) complete probability measurable space then $(X, \mathcal{F}, t_m)$ is $\mathcal{F}$-complete[7]. A mapping $f\ \Omega \times M \to M$ is a random operator if for any $x \in M$, $y(w) = f(w, x)$ is a M-valued random variable. A random operator f is called d-continuous if for each $w \in \Omega$, $f(w,.)$ is continuous in the topology induced by the metric d. A random variable $\xi(w) \in X$ is a random fixed point of the random operator f if $\xi(w) = f(w, \xi(w))$ for all $w \in \Omega$.

It is obvious that all results of Theorem 1 can be applied to the E-space.

Theorem 3—Let (M, d) be a separable complete metric space, (Ω, K, P) a complete probability measurable space and $(X, \mathcal{F}, t_m)$ a E-space over (M, d). Suppose that $f: \Omega \times M \to M$ is the mapping such that for some $m \in N$, f^m is d-continuous random operator, let $\{g_n\}_{n\in N}$, $g_n: \Omega \times M \to M$ be a family of random operators

such that $g_n\ f^{m-1}(\Omega \times M) \subset f^m(\Omega \times M)$ and $g_n f = f g_n$ for all $n \in N$, and let there exists a sequence $\{m_n\}_{n\in N}$, $m_n \in N$, such that for all $n, k \in N$ and all $x, y \in f^{m-1}(\Omega \times M)$

$$F_{g_n^{m_n}(w, x(w(w)),\ g_k^{m_k}(w, y(w))}(\epsilon)$$

$$> \min_{u, v \in f(w, x(w)), f(w, y(w)), g_n^{m_n}(w, x(w)), g_k^{m_k}(w, y(w))} F_{u,v}(\phi(\epsilon)).$$

If there exists $x_1 \in f^{m-1}(\Omega \times M)$ such that the sequence $\{y_n\}_{n\in N}$ formed by $x_n(w) = g_n^{m_n}(w, x_n(w)) = f(w, x_{n+1}(w))$ is probabilistic bounded, then there exists a unique common fixed point for the family $\{g_n\}_{n\in N}$ and f.

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