Indian J. pure appl. Math., **17** (4) : 476-480, April 1986

FIXED POINT THEOREMS FOR THREE MAPS IN UNIFORM SPACES

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(*Received* 10 *June* 1985)

Fixed point theorems is a Hausdorff uniform space are proved by considering a contractive type conditions for a triplet of mappings.

Let (X, U) be a uniform space. A family $\{d_\lambda : \lambda \in I\}$ of pseudometrics on X is called an associated family for the uniformity U on X if the family $B = \{V_{(\lambda,r)} : \lambda \in I, r > 0\}$ is a subbase for the uniformity U, where $V_{(\lambda,r)} = \{(x, y) \in X \times X: d_\lambda (x, y) < r\}$. We may assume B itself to be a base for U, by adjoining the finite intersection of members of B, if necessary. The corresponding family of pseudometrics is called an augmented associated family for U. An associated family and an augmented associated family for U will be denoted by D and D^* respectively. For details, one can refer to Tarafdar[15] and Thron[16].

A triplet of mappings (P, Q, T) of a uniform space X is called quasi-contraction if,

$$d_\lambda (Px, Qy) \leqslant q_\lambda \max \{d_\lambda (Tx, Ty), d_\lambda (Px, Tx), d_\lambda (Qy, Ty), \tfrac{1}{2} [d_\lambda (Px, Ty) + d_\lambda (Qy, Tx)]\} \quad ...(1.1)$$

for all $x, y \in X, \lambda \in I, q_\lambda < 1$.

A triplet (P, Q, T) is called a generalized contraction if, for all $x, y \in X, \lambda \in I$,

$$d_\lambda (Px, Qy) \leqslant a_\lambda d_\lambda (Tx, Ty) + b_\lambda d_\lambda (Px, Tx) + c_\lambda d_\lambda (Qy, Ty) + e_\lambda d_\lambda (Px, Ty) + f_\lambda d_\lambda (Qy, Tx) \quad ...(1.2)$$

$0 < a_\lambda + b_\lambda + c_\lambda + e_\lambda + f_\lambda < 1$ and $e_\lambda = f_\lambda$.

Clearly a generalized contraction triplet is always a quasi-contraction (see Park[8], for discussion on single and two mappings). Mapping condition (1.1) occur for metric space in Singh and Singh[14] and (1.2) occur for uniform space in Singh and Mishra[13].

In the present note, first we present a common fixed point theorem under (1.1). Then we study the relationship between the convergence of three sequences of mappings and the convergence of their common fixed points.

Theorem 1—Let (X, U) be a sequentially complete Hausdorff uniform space and $\{d_\lambda : \lambda \in I\} = D^*$. Suppose that P, Q, T are mappings from X to itself such that $PT = TP$, $QT = TQ$ and $P(X) \cup Q(X) \subseteq T(X)$. If T is continuous and the triplet (P, Q, T) is a quasi-contraction, then P, Q and T have a unique common fixed point.

PROOF : Pick $x_0 \in X$. Let $\{x_n\}$ be a sequence in X such that $Tx_{2n+1} = Px_{2n}$, $Tx_{2n+2} = Qx_{2n+1}$, $n = 0, 1, 2, \ldots$ This is always possible as $P(X) \cup Q(X) \subseteq T(X)$. Now from (1.1) we have $d_\lambda (Tx_{n+1}, Tx_n) \leqslant q_\lambda d_\lambda (Tx_n, Tx_{n-1})$, so that $\{Tx_n\}$ is a Cauchy sequence and converges to some $z \in X$. Consequently $Tx_{2n} \to z$, $Px_{2n} \to z$ and $Qx_{2n+1} \to z$. By the continuity of T and the fact that $PT = TP$, it follows that $TTx_{2n} \to Tz$ and $TTx_{2n} \to Tz$ and $PTx_{2n} = TPx_{2n} \to Tz$. Taking $x = Tx_{2n}$ and $y = z$ in (1.1) and making $n \to \infty$, we get $d_\lambda (Px, Qz) \leqslant q_\lambda d_\lambda (Qz, Tz)$ so that $Tz = Qz$. Similarly, $Tz = Pz$. Again, taking $x = x_{2n}$ and $y = z$ in (1.1) we can show that $z = Qz = Tz = Pz$. The uniqueness of z easily follows.

As a slight generalization of the above theorem, we have the following :

Theorem 2—Let (X, U) be a sequentially complete Hausdorff uniform space and $\{d_\lambda : \lambda \in I\} = D^*$. Suppose that P, Q and T are mappings from X to itself such that $PT = TP$ and $QT = TQ$. If there exists positive integers p, q and t such that T^t is continuous, $P(X) \cup Q(X) \subseteq T^t(X)$ and (P^p, Q^q, T^t)-triplet is a quasi-contraction, then P, Q and T have a unique common fixed point.

PROOF : It may be completed on the lines of Singh[10] (Cor., p. 93).

Now we have the convergence theorem.

Theorem 3—Let (X, U) be a sequentially complete Hausdorff uniform space and $\{d_\lambda : \lambda \in T\} = D^*$. Let $\{P_n\}$, $\{Q_n\}$ and $\{T_n\}$ be sequence of mappings from X to itself such that the triplet (P_n, Q_n, T_n) is a quasi-contraction with a control constant fixed $(q_\lambda)_n$ for a fixed λ and for each $n = 1, 2, \ldots$, If P, Q and T: $X \to X$ are pointwise limit of P_n, Q_n and T_n, respectively, and if $(q_\lambda)_n \to q_\lambda < 1$, then (P, Q, T) is a quasi-contraction with q_λ. Furthermore, the sequence of the unique common fixed point u_n of P_n, Q_n and T_n converges to the unique common fixed point u of P, Q and T.

PROOF : For any $x, y \in X$, we hove, for $x \neq y$,

$$d_\lambda (P_x, Q_y) \leqslant d_\lambda (Px, P_n x) + d_\lambda (P_n x, Q_n y) + d_\lambda (Q_n y, Q_y)$$

(equation continued on p. 478)

$$\leqslant [d_\lambda (P_n x, P x) + d_\lambda (Q_n y, Q y)$$

$$+ (q_\lambda)_n, \max \{d_\lambda (T_n x, T_n y), d_\lambda (P_n x, T_n x), d_\lambda (Q_n y, T_n y),$$

$$\left[d_\lambda \left(P_n x, T_n y\right) + d_\lambda \left(Q_n y, T_\lambda x\right)\right]/2\Big\}.$$

Since $P_n x \to Px$, $Q_n y \to Q y$, $T_n x \to Tx$ and $T_n y \to T y$ for $x \neq y$, and also q_{λ_n} $\to q_\lambda < 1$ as $n \to \infty$, we have $d_\lambda (Px, Qy) \leqslant [q_\lambda \max \{d_\lambda (T x, Ty), d_\lambda (Px, Tx)]$, $d_\lambda (Qy, Ty), [d_\lambda (Px, Ty) + d (Qy, T xq]/2\}$.

Therefore, (P, Q, T) is a quasi-contraction, and P, Q and T have a unique common fixed point u since X is complete sequentially.

Let u_n be the common fixed point of P_n, Q_n and T_n for each n. Then

$$d_\lambda (u_n, u) \leqslant d_\lambda (P_n u_n, Q_n u) + d_\lambda (Q_u u, u)$$

$$\leqslant q_\lambda \max. \{d_\lambda (T_n u_n, T_n u), d_\lambda (P_n u_n, T_n u_n),$$

$$d_\lambda (Q_n u, T_n u), [d_\lambda (P_n u_n, T_n u) + d_\lambda (Q_n u, T_{nn} u)]/2\}$$

$$+ d_\lambda Q_n u, u).$$

$$\leqslant q_\lambda \max. \{d_\lambda (u_n, T_n u), d_\lambda (Q_n u, T_n u),$$

$$\tfrac{1}{2} [d_{\alpha\lambda}(u_n, T_n u) + d_\lambda (Q_n u, T u_n)]\}$$

$$+ d_\lambda (Q_n u, u).$$

W will take the terms inside the crooked bracket one by one and prove the result. If the term is $d_\alpha (u_n, T_n u)$,

$$d_\lambda (u_n, u) \leqslant q_\lambda (d_\lambda (u_n, u) + d_\lambda (u, T_n u)) + d_\lambda \left(Q_n u, u \right)$$

$$d (u_n, u) \leqslant \frac{q_\lambda}{1-q_\lambda} d_\lambda (Tu, T_n u) + \frac{1}{1 - q_\lambda} d (Q_n u, Q u).$$

Since $Q_n \to Q$ and $T_n \to T$, pointwise it follows that, as $n \to \infty$,

$d_\lambda (u_n, u) \to 0$, i. e. $u_n \to u$. If the term is $d (Q_n u, T_n u)$,

then

$$d_\lambda (u_n, u) \leqslant q_\lambda \{d_\lambda (Q_n u, u) + d_\lambda (T_n u, u)\} + d_\lambda (Q_n u, u)$$

$$\leqslant (1 + q_\lambda) d_\lambda (Q_n u, Qu) + q_\lambda d_\lambda (T_n u, Tu), \text{ so that as } n \to \infty,$$

$u_n \to u$. Similarly, for the last term in the bracket, we can prove that $u_n \to u$.

Remarks 1 : For the triplet-(P, Q, T) a generalized contraction, we have the results of Singh and Mishra[13].

2. For $P = Q$, (1.1) and (1.2) give definitions introduced by Singh[9,11]. In fact, our Theorem 3 under (1.1) gives result whichis extension of the results of Singh[11] under (1.2) in metric space. Singh's results[11] were further extension of the results of many authors. Dube and Singh[3], Singh[12].

3. For $Tx = x$, our Theorem 3 is an extension of the Theorem 4, Park[8] which in turn extended. Theorem 4 of Mukherjee[8] under generalized condition with $a_\lambda = e_\lambda = f_\lambda = 0$, $b_\lambda = c_\lambda$ for all $x, y \in X$ —a metric space, $x \neq y$. For a single mapping, Park[8] also obtained extension of result of Achari[1] for quasi-contraction g satisfying condition.

$$d(gx, gy) \leqslant \alpha \max \{d(x, y), [d(x, gx) + d(y, gy)]/2.$$

$[d(x, gy) + d(y, gx)]/2\}$ which was a further extension of Bonsall[2] and Nadler[7] for maps g satisfying condition (1.2) with $b_\lambda = c_\lambda = e_\lambda = f_\lambda = 0$.

4. For $P = Q$, $Tx = x$, for every $x \in X$, our Theorem 3 is an improvement of the result of Mishra[5] under condition (1.2). Even result of Isèki[4] is generalized.

ACKNOWLEDGEMENT

The author is indebted to Professor S.L. Singh for his valuable help in the preparation of this paper.

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