Indian J. pure appl. Math., 17 (4) : 481-484, April 1986

M-PROJECTIVELY FLAT SASAKIAN MANIFOLDS

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(*Received* 5 *October* 1983; *after revision* 11 *November* 1985)

In this paper some properties of *M*-projective curvature tensor have been studied in a Sasakian manifold.

1. INTRODUCTION

Let us consider an $n\ (=2m+1)$ dimensional differentiable manifold V_n of differentiability class C^∞, a C^∞ – vector valued linear function F, a C^∞ – vector field T and a C^∞ 1-form A satisfying

$$\text{(a)}\quad A(T)=1, \qquad \text{(b)}\quad \bar{X}+X=A(X)T,$$

$$\text{(c)}\quad \bar{T}=0, \qquad \text{(d)}\quad A(\bar{X})=0 \qquad \ldots(1.1)$$

where $\bar{X}$ def $F(X)$ for arbitrary vector field X. Then V_n is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure.

Let the almost contact manifold V_n be endowed with the metric tensor g satisfying

$$g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)\ A(Y). \qquad \ldots(1.2)$$

Then V_n is called an almost contact metric manifold. If in an almost contact metric manifold

$$2'F(X, Y) = (D_X A)(Y) - (D_Y A)(X) \qquad \ldots(1.3)$$

where $'F(X, Y) = g(\bar{X}, Y)$ and D is a Riemannian connection, then V_n is called an almost Sasakian manifold. An almost Sasakian manifold is said to be Sasakian manifold if T is a Killing vector

$$(D_X A)(Y) + (D_Y A)(X) = 0. \qquad \ldots(1.4)$$

In a Sasakian manifold we have[3]

$$\text{(a)}\quad K(T, Y, Z) = Tg(Y, Z) - YA(Z),$$

$$\text{(b)}\quad K(X, Y, T) = XA(Y) - YA(X),$$

$$\text{(c)}\quad \mathrm{Ric}\,(X, T) = 2m\,A\,(X),$$

$$\text{(d)}\quad r\,(T) = 2mT, \qquad \ldots(1.5)$$

where K, Ric and r are the curvature tensor, the Ricci tensor and the Ricci map[4] respectively. Also, in a Sasakian manifold, we have

$$\text{(a)}\quad {}'K\,(T, X, Y, T) = g\,(\bar{X}, \bar{Y}) = g\,(X, Y) - A\,(X)\,A\,(Y)$$

$$\text{(b)}\quad {}'K\,(T, X, Y, Z) = A\,(Z)\;g\,(X, Y) - g\,(X, Z)\,A\,(Y) \qquad \ldots(1.6)$$

where ${}'K$ is the curvature tensor of type (0, 4).

2. M-Projective Curvature Tensor

The M-projective curvature tensor M in a Sasakian manifold is given by

$$M\,(X, Y, Z) = K\,(X, Y, Z) - \frac{1}{4m}\,\{X\,\mathrm{Ric}\,(Y, Z) - Y\,\mathrm{Ric}\,(X, Z) + r\,(X)\,g\,(Y,Z) - r\,(Y)\,g\,(X, Z)\}.$$

It can be easily proved that

$$\text{(a)}\quad M\,(T, Y, Z) = \frac{T}{2}\left\{g\,(Y, Z) - \frac{1}{n-1}\,\mathrm{Ric}\,(Y, Z)\right\} - \frac{1}{2}A\,(Z)\left(Y - \frac{1}{n-1}\,r\,(Y)\right)$$

$$\text{(b)}\quad M\,(Y, P, T) = \frac{1}{2}A\,(P)\left(Y - \frac{1}{n-1}\,r\,(Y)\right) - \frac{1}{2}A\,(Y)\left(P - \frac{1}{n-1}\,r\,(P)\right). \qquad \ldots(2.1)$$

Now, the following theorem will be proved :

Theorem 2.1—A M-projectively recurrent Sasakian manifold is M-projectively flat if and only if it is an Einstein manifold.

Proof : If the Sasakian manifold is M-projectively recurrent then we have

$$(D_X\,M)\,(Z, P, Q) = B\,(X)\,M\,(Z, P, Q). \qquad \ldots(2.2)$$

From (2.2) we have

$$K\,(X, Y, M\,(Z, P, Q)) - M\,(K\,(X, Y, Z), P, Q) - M\,(Z, K\,(X, Y, P), Q) - M\,(Z, P, K\,(X, Y, Q)) = \{(D_X\,B)\,(Y) - (D_Y\,B)\,(X)\}\,M\,(Z, P, Q). \qquad \ldots(2.3)$$

Putting T for X and Z in the above equation and using (1.5), (1.6) and (2.1), we get

$$M(Y, P, Q) + \frac{1}{2} A(P) T\left\{g(Y, Q) - \frac{1}{n-1} \operatorname{Ric}(Y, Q)\right\}$$

$$- \frac{1}{2} A(P) A(Q)\left\{Y - \frac{1}{n-1} r(Y)\right\}$$

$$- \frac{1}{2} Y\left\{g(P, Q) - \frac{1}{n-1} \operatorname{Ric}(P, Q)\right\}$$

$$+ \frac{1}{2} g(Y, Q)\left\{P - \frac{1}{n-1} r(P)\right\}$$

$$= \{(D_T B)(Y) - (D_Y B)(T)\}\left\{\frac{T}{2}(g(P, Q)\right.$$

$$\left. - \frac{1}{n-1} \operatorname{Ric}(P, Q)) - \frac{1}{2} A(Q)\left(P - \frac{1}{n-1} r(P)\right\}\right. . \quad \ldots(2.4)$$

If the Sasakian manifold is an Einstein manifold, $M(Y, P, Q) = 0$. Conversely, let the Sasakian manifold be M-projectively flat. Then it can be easily proved that it is an Einstein manifold.

The author proved the following theorem in Sasakian manifold[1] :

Theorem—In a Sasakian manifold if

$$(D_V W)(X, Z, U) = 0, \quad (D_Y M)(X, Z, U) = 0$$

then

$$'W(X, Z, U, V) = 2' M(X, Z, U, V)$$

where

$$'M(X, Y, Z, V) = g(M(X, Y, Z), V)$$

and $'W$ is the Weyl projective curvature tensor.

Taking account of Theorem 2.1 and the fact that a projectively recurrent Sasakian manifold is projectively flat[2] the above theorem may be restated as follows:

Theorem 2.2—In a Sasakian manifold if

$$(D_Y W)(X, Z, U) = 0, \quad (D_Y M)(X, Z, U) = 0,$$

then

$$'W(X, Z, U, V) = 'M(X, Z, U, V) = 0.$$

Theorem 2.3—Let V_n be a Sasakian manifold. Then the following conditions are equivalent :

(i) V_n is M-projectively symmetric,

(ii) V_n is of constant curvature 1,

(iii) $R(X, Y) R = 0$,

where the curvature transformation acts on the tensor algebra as a derivation[5].

PROOF : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obviously. Now, we take

$$R(X, Y) R = 0 \Rightarrow R(X, T) R = 0,$$

which is equivalent to

$$R(X, T, R(V, Z, Q)) - R(R(X, T, V), Z, Q) - R(V, R(X, T, Z), Q) - R(V, Z, R(X, T, Q)) = 0. \quad \ldots(2.5)$$

Putting T for V and Q in (2.5) and using (1.1), (1.5) and (1.6) we obtain

$$R(T, Z, X) = g(X, Z) T - A(X) Z. \quad \ldots(2.6)$$

Putting T for V in (2.5) and using (2.6), we get

$$R(X, Z, Q) = g(Q, Z) X - g(X, Q) Z$$

which shows that V_n is of constant curvature.

(ii) $\Rightarrow$ (i) obviously. This completes the proof.

ACKNOWLEDGEMENT

The author is grateful to the referee for pointing out certain oversights in our original version.

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