Indian J. pure appl. Math., 17 (4): 485-487, April 1986

THE CENTRE OF A METACYCLIC GROUP

M. A. EL-GENDY

Physics and Mathematics Department, Alexandria University, Alexandria, Egypt

(*Received* 2 *February* 1985; *after revision* 24 *October* 1985)

Various problems concerning metacyclic groups had been dealt with by many authors[1,3,4].

The aim of this paper is to characterize the centre of finite metacyclic groups.

We begin with the following definitions :

Definition 1—A group G is called metacyclic if it has a cyclic normal subgroup which yields a cyclic factor.

A metacyclic group G can be presented by

$$G = < x, y \mid x^m, y^{-1}\ x\ y = x^r,\ \ y^n\ = x^s >, \quad \text{where } m, n,$$

r and s are natural numbers and $r, s \leqslant m,\ r^n \equiv 1 \pmod{m}$,

$$r\ s \equiv r \pmod{m}.$$

We must note that for any metacyclic group, the order d_r of r modulo m [ref. 2] divides n. Also $d_r \neq 1$, for if $d_r = 1$, then $r \equiv 1 \pmod{m}$ i. e.

$r > m$, which contradicts the definition of a metacyclic group.

Let G be a metacyclic group with the specified parameters m and r. Since the value of the parameter s, $s < m$, should satisfy the congruence $r\ s \equiv s \pmod{m}$, then s must take the form

$$s = \frac{\lambda m}{(m, r-1)}$$

where $1 \leqslant \lambda \leqslant (m, r-1)$. Hence the least value that s can assume is $\frac{m}{(m, r-1)}$ which we denote by s_0.

Theorem—An element $y^a\ x^b\ y^c$ is in the centre $Z(G)$ of a metacyclic group

$$G = < x, y\ \ x^n = e,\ y^{-1}\ x\ y = x^r, y^n = x^s >$$

where $r^n \equiv 1 \pmod{m}$ and $rs \equiv s \pmod{m}$ if and only if (i) $a + c \equiv o \pmod{d_r}$, (ii) $b \equiv o \pmod{s_0}$.

PROOF : Since $r^{d_r} \equiv 1 \pmod{m}$, then

$$y^{-d_r} x\, y^{d_r} = x.$$

Therefore,

$$x\, y^{d_r} = y^{d_r} x.$$

Also,

$$y\, y^{d_r} = y^{d_r} y.$$

Hence y^{d_r} commutes with x and y, and so $y^{d_r} \in Z(G)$.

Now, $z = y^a x^b y^c \in Z(G)$, $a, c \geqslant 0$ iff

$$xz = zx \text{ and } y\, z = zy.$$

But $xz = zx$ is the same as $xy^a x^b y^c = y^a x^b y^c x$.

Since $y^{-b} x^a y^b = x^{ar^b}$ and $y^b x^a y^{-b} = x^{ar^{(n-1)b}}$ then

$$x\, y^a = y^a x^{r^a} \text{and } y^c x = x^{r^{(n-1)c}} y^c,$$

and so $y^a x^{r^a} x^b y^c = y^a x^b x^{r^{(n-1)c}} y^c$.

Therefore, $x\, z = z\, x$ iff $b + r^a = b + r^{(n-1)c} \pmod{m}$

or $r^a = r^{nc} r^{-c} \pmod{m}$

i. e., $r^a r^c = r^{nc} \pmod{m}$.

But $r^{nc} = 1 \pmod{m}$.

Consequently, $r^{a+c} = \pmod{m}$.

Hence $d_r \mid (a + c)$ or $a + c \equiv O \pmod{d_r}$.

On the other hand, $yz = zy$ is the same as

$$yy^a x^b y^c = y^a x^b y^c y \text{ or } y^{a+1} x^b y^c = y^{a+1} x^{br} y^c.$$

Therefore, $yz = zy$ iff $b \equiv br \pmod{m}$,

$$\text{i. e. } b = \frac{tm}{(m, r-1)} = ts_0.$$

Therefore, $b \equiv O \pmod{s_0}$, as required.

Corollary—If $(m, r-1) = 1$. Then the centre $Z(G)$ of a finite metacylic group G is a cyclic subgroup generated by y^{d_r}.

PROOF : Let $y^a x^b y^c \in Z(G)$. Since $(m, r-1) = 1$ then $s = m = s_0$. From the previous theorem we have $b = tm$ for some integer t.

Also the congruence $a + c \equiv O \pmod{d_r}$ implies that

$$u\, d_r = a + c, \text{ for some integer } u.$$

Now

$$y^a x^b y^c = y^a x^{tm} y^c = y^{a+c} = y^{ud_r}.$$

Hence $Z(G) = < y^{d_r} >$, as required.

REFERENCES

1. F. R. Beyl, *Proc. Am. Math. Soc.* **40** (1973), 413-18.
2. G. H. Hardy, and E. M. Wright, *An Introduction to the Theory of Numbers.* 4th edition, Oxford University Press, New York, 1960.
3. J. S. Rose, Finite Metacyclic groups, all extensions of which split, *Proc. R. Soc. Edinburgh*, **79A** (1977), 285-91.
4. H. J. Zassenhaus, *The Theory of Groups I.* Chelsea, New York 1974.