Indian J. pure appl. Math., **17** (4) : 488-496, April 1986

ON MIXED VOLTERRA-FREDHOLM TYPE INTEGRAL EQUATIONS

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(*Received* 13 *September* 1984; *after revision* 10 *December* 1985)

In this paper we study the existence and uniqueness of the solutions of mixed Volterra-Fredholm type integral equations which arises in various physical and biological problems.

1. INTRODUCTION

Heard[6] has studied the existence and other properties of the solutions of the following integrodifferential equation

$$\frac{\partial u}{\partial t}(t, x) = \int_a^b k(x, y) f(t, y, u(t, y))\, dy, \quad u(0, x) = u_0(x) \qquad \ldots(1)$$

where a and b are finite real numbers and $k(x, y)$ is a Lebesgue square summable function defined on the square $a \leqslant x, y \leqslant b$. The special case of equation (1) when $k(x, y)$ is continuous and $f(t, y, u) \equiv u$ was studied by Volterra[16]. In this paper we consider the following general mixed Volterra-Fredholm type integral equation

$$u(t, x) = f(t, x) + \int_0^t \int_a F(t, x, s, y, u(s, y))\, dy\, ds \qquad \ldots(2)$$

where u is an unknown function, $u, f \in C[D, R^n]$, $F \in C[D^2 X R^n, R^n]$ in which Ω is a bounded domain in the n dimensional Euclidean space R^n with convenient norm $\|.\|$, $D = [0, T]$ $X\, \Omega$ and $T > 0$ is finite but can be arbitrarily large. The integral equation (2) is of Volterra type with respect to t and of Fredholm type with respect to x. Equations of the type (2) arise, for example in the theory of nonlinear parabolic boundary value problems and in certain epidemic models[3,14].

The purpose of this paper is to study the basic problem of existence and uniqueness of the solutions of equation (2). The well known Banach fixed point theorem is used to establish our results. We also discuss the existence and uniqueness of the solutions of some specific physical and biological systems which are governed by the equation (2). In fact our study of equation (2) is motivated by its occurrence in various applied problems.

2. Main Result

Let S be a space of those functions $\phi : D \to R^n$ which are continuous in D and fulfill the condition

$$\| \phi (t, x)\| = O (\exp (\mu (t + \| x\|))), \ (t, x) \in D \qquad ...(3)$$

where $\mu > 0$ is a constant. In the space S we define the norm Czerwik[2], Pachpatte[11,12]

$$| \phi | = \sup_D [\|\phi (t, x) \| \exp (- \mu (t + \|x\|)]. \qquad ...(4).$$

It is easily seen that S with norm defined in (4) is a Banach space[2]. We note that the condition (3) implies that there exists a nonnegative constant M such that $\|\phi (t, x)\| \leqslant M \exp (\mu (t + \|x\|)), \ (t, x) \in D$. Using this fact in (4) we observe that

$$| \phi | \leqslant M. \qquad ...(5)$$

Our main result is the following theorem which gives sufficient conditions for the existence and uniqueness of the solutions of equation (2).

Theorem 1—Suppose that

(H_1) there exists a nonnegative continuous function $h (t, x, s, y)$ defined on D^2 such that

$$\|F (t, x, s, y, u_1) - F (t, x, s, y, u_2) \| \leqslant h (t, x, s, y) \| u_1 - u_2\|$$

for $(t, x, s, y, u_i) \in D^2 \ X \ R^n$, $i = 1, 2$ and

$$\int_0^t \int_a h (t, x, s, y) \exp (\mu (s + \|y\|)) \, dy \, ds \leqslant Q \exp (\mu (t + \|x\|))$$

where Q is a nonnegative constant ;

(H_2) there exists a constant $N > 0$ such that

$$\|f (t, x)\| + \int_0^t \int_a \| F (t, x, s, y, 0)\| \, dy \, ds \leqslant N \exp (\mu (t + \|x\|)).$$

If $Q < 1$, then there exists exactly one solution $u \in S$ of eqn. (2) given as the limit of the successive approximations.

Proof : For $u \in S$, we define the operator $A (u)$ by the right side of the equation (2). Now we shall prove that $A (u)$ maps S into itself. Evidently $A (u)$ is continuous in D and $A (u (t, x)) \in R^n$ for $(t, x) \in D$. We verify that (3) is fulfilled. Using assumptions (H_1) and (H_2) we obtain

$$\|A(u(t,x))\| \leqslant \int_0^t \int_a \|F(t,x,s,y,u(s,y)) - F(t,x,s,y,0)\| dy\, ds$$

$$+ \|f(t,x)\| + \int_0^t \int_a \|F(t,x,s,y,0)\| dy\, ds$$

$$\leqslant |u| \int_0^t \int_a h(t,x,s,y) \exp(\mu(s + \|y\|)) dy\, ds$$

$$+ N \exp(\mu(s + \|x\|))$$

$$\leqslant [MQ + N] \exp(\mu(t + \|x\|)).$$

From this it follows that $A(u) \in S$.

Now we verify that the operator $A(u)$ is a contraction map. We assume that $u, z \in S$, then from assumption (H_1) we have

$$\|A(u(t,x)) - A(z(t,x))\| \leqslant \int_0^t \int \|F(t,x,s,y,u(s,y))$$

$$- F(t,x,s,y,z(s,y))\| dy\, ds$$

$$\leqslant |u - z| \int_0^t \int_a h(t,x,s,y)$$

$$\exp(\mu(s + \|y\|)) dy\, ds$$

$$\leqslant Q |u - z| \exp(\mu(t + \|x\|)).$$

Consequently, we have

$$|A(u) - A(z)| \leqslant Q |u - z|.$$

Now an application of the well known Banach fixed point theorem[10], there exists a unique solution $u \in S$ of eqn. (2) and it is given as the limit of the successive approximations. This completes the proof of the theorem.

We note that the result obtained in Theorem 1 can be very easily extended to more general mixed Volterra-Fredholm integral equation of the form

$$u(t,x) = f(t,x) + \int_0^t P(t,x,s,y,u(s,y)) ds + \int_a B(t,x,s,y,u(s,y)) dy$$

$$+ \int_0^t \int_a F(t,x,s,y,u(s,y)) dy\, ds \qquad \ldots(6)$$

under some suitable conditions on the functions involved in (6) in view of the assumptions (H_1) and (H_2) on the functions involved in eqn. (6). The precise formulation of existence and uniqueness result for eqn. (6) is similar to that of our Theorem 1 and we omit it here. Equation (6) is of more general type and contains as a special case the integrodifferential equation studied by Liberman[9] which in itself contains as a special case the equation (1) studied by Heard[6].

3. Some Applications

In this section we discuss the existence and uniqueness of solutions of certain nonlinear parabolic differential equations and mixed Volterra-Fredholm integral equations occurring in specific physical and biological problems.

Consider the following nonlinear parabolic differential system

$$\frac{\partial u}{\partial t} - Lu = F(t, x, u), \quad (t \in (0, T], x \in \Omega) \qquad \ldots(7)$$

with the given boundary and initial conditions

$$\frac{\partial u}{\partial \nu} + b(x)\, u = C_0(x), (t \in (0, T], \ x \in \partial\Omega) \qquad \ldots(8)$$

$$u(0, x) = u_0(x), (x \in \Omega), \qquad \ldots(9)$$

where $u \in C[R^+ \ X \ \bar{\Omega}, R^n]$, $F \in C[R^+ X \bar{\Omega} X R^n, R^n]$, $R^+ = [0, \infty)$, Ω is a bounded domain in R^n, $\partial\Omega$ is the boundary of Ω and $\bar{\Omega}$ is the closure of Ω, $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$ and L is the uniformly elliptic operator in the form

$$L = \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(t, x) \frac{\partial}{\partial x_i}. \qquad \ldots(10)$$

It is assumed that $b(x) \geqslant 0$ on $\partial\Omega$, the coefficients of L and the first partial derivatives of a_{ij} are Hölder continuous in $R^+ X \bar{\Omega}$ (of exponent α, $0 < \alpha < 1$), the matrix (a_{ij}) is symmetric positive definite in $R^+ X \bar{\Omega}$, F is Hölder continuous in (t, x, u) in every bounded subset of $R^+ X \bar{\Omega} X R^n$, the boundary $\partial\Omega$ is of class $C^{2+\alpha}$, $b \in H^{1+\alpha}(\partial\Omega)$, $C_0 \in H^{1+\alpha}(C_0 T] X \partial^a u_0 \in H^{2+\alpha}(\bar{\Omega})$ and finally u_0 satisfies the boundary condition (8) at $t = 0$, where $H^{1+\alpha}(\partial\Omega)$, $H^{2+\alpha}(\bar{\Omega})$ $H^{1+\alpha}(C_0, T] X \partial^a)$ are the function spaces in the sense of Friedman[5] (see also ref. 8). The equations of the form (7) – (8) represents some concrete models in the area of reaction diffusion equations and in recent years there has been increasing attention in the study of such equations, see Pao[15] and the references given therein.

The boundary value problem (7) −(9) can be written in the following equivalent form of mixed Volterra-Fredholm integral equation

$$u(t,x) = \int_0^t \int_a G(t, x; s, y)\, F(s, y, u(s, y))\, dy\, ds + \int_0^t \int_{\partial a} G(t,x;s,y)$$

$$\psi(s, y)\, dy\, ds + \int_a G(t, x; 0, y)\, u_0(y)\, dy \qquad \ldots(11)$$

where G is the fundamental solution of $\frac{\partial v}{\partial t} - L v = 0$, (see, Friedman[5], p. 144 ; Ladyzenskaja[8], p. 395). The second integral in (11) is a single layer potential with the density $\psi(t, x)$ which can be determined from the following Volterra type integral equation

$$\psi(t, x) = \int_0^t \int_{\partial a} R(t, x; s, y)\, \psi(s, y)\, dy\, ds + H(t, x, u(t, x)),$$

where

$$R(t, x; s, y) = 2\left[\frac{\partial G}{\partial \nu}(t, x; s, y) + b(x)\, G(t,x;s,y)\right]$$

and $H(t,x, u(t,x))$ is given by

$$H(t, x, u(t, x)) = \int_a R(t, x; 0, y)\, u_0(y)\, dy + \int_0^t \int_a R(t, x; s, y)$$

$$\times F(s, y, u(s, y))\, dy\, ds - C_0(x).$$

In fact ψ is given by (see Friedman[5], p. 145)

$$\psi(t, x) = H(t, x, u(t, x)) + 2\int_0^t \int_{\partial a} R(t, x; s, y)\, H(s, y, u(s, y))\, dy\, ds.$$

For the details of the derivation of eqn. (11) we refer the readers to [reference 5 and 8].

We consider eqn. (11) under the following hypotheses.

(H_3) There exists a nonnegative continuous function $g(t, x)$ defined on D such that

$$\|F(t, x, u_1) - F(t, x, u_2)\| \leqslant g(t, x)\, \|u_1 - u_2\|,$$

for

$$(t, x, u_i) \in D \times R^n, (i = 1, 2).$$

(H_4) There exist nonnegative constants Q_1, Q_2, Q_3 such that

$0 \leqslant Q_1 + Q_2 Q_3 < 1$ and

$$\int_0^t \int_{\partial\Omega} \|G(t, x; s, y)\| g(s, y) \exp(\mu(s + \|y\|)) \, dy \, ds \leqslant Q_1 \exp(\mu(t + \|x\|)),$$

$$\int_0^t \int_{\partial\Omega} \|G(t, x; s, y)\| [\exp(\mu(s + \|y\|)) + 2 \int_0^s \int_{\partial\Omega} \|R(s, y; \tau, z)\| \|\exp(\mu(\tau + \|z\|)) \, dz \, d\tau] \, dy \, ds \leqslant Q_2 \exp(\mu(t + \|x\|))$$

$$\int_0^t \int_{\partial\Omega} \|R(t, x; s, y)\| g(s, y) \exp(\mu(s + \|y\|)) \, dy \, ds \leqslant Q_3 \exp(\mu(t + \|x\|).$$

(H_5) There exist nonnegative constants N_1, N_2 such that

$$\int_\Omega \|G(t, x; 0, y)\| \, \|u_0(y)\| \, dy + \int_0^t \int_{\partial\Omega} \|G(t, x; s, y)\| \, \|F(s, y, 0)\| \, dy \, ds \leqslant N_1 \exp(\mu(t + \|x\|))$$

$$\int_\Omega \|R(t, x; 0, y)\| \, \|u_0(y)\| \, dy + \int_0^t \int_{\partial\Omega} \|R(t, x; s, y)\| \, \|F(s, y, 0)\| \times dy \, ds + \|C_0(x)\| \leqslant N_2 \exp(\mu(t + \|x\|)).$$

We are now in a position to establish the following :

Theorem 2—Suppose that the hypotheses $(H_3) - (H_5)$ hold. Then there exists a unique solution $u \in S$ of equation (11) given as the limit of the successive approximations.

The details of the proof of this theorem are analogous to Theorem 1 given in section 2 with suitable modifications. We omit the details.

We note that the authors[13-15] have studied the existence of maximal and minimal solutions of more general versions of the boundary value problem (7) – (9) by using the method of upper and lower solutions. In the above discussion we do not aim at generality. Here our Theorem 2 asserts the existence and uniqueness of the solutions of

problem (7) – (9) by using different assumptions and technique from those of given earlier[13–15].

In the following example, we describe, in brief Diekmann's[2] model of spatio-temporal development of an epidemic which leads to a nonlinear integral equation of mixed Volterra-Ferdholm type considered in (2). The notations, definitions and the method of derivation of equation is due to Diekmann[3].

Example (Diekmann[3]). Consider a population living in a habitat Ω, a closed subset of R^n, and susceptible to some contagious disease. Let $S(t, x)$ and $I(t, x)$ denote the density of susceptibles and infectives respectively at time t and position x. Let $i(t, \tau, x)\, d\tau$ be the density of infectives which were infected some time between $t - \tau$ and $t - \tau - d\tau$. Then

$$I(t, x) = \int_0^\infty i(t, \tau, x)\, d\tau. \qquad \ldots(12)$$

Suppose that the population size is large, so that one can contider S, I and i are sufficiently smooth real-valued functions of their arguments. Let the infectivity $B = s$ $B(t, x)$ be defined as the rate at which susceptibles become infective. Assume that (*i*) the disease induces permanent immunity, so the transition from I to S does not occur, (ii) a nonnegative function $A = A(\tau, x, \xi)$ is given such that

$$B(t, x) = \int_0^\infty \int_\Omega i(t, \tau, \xi)\, A(\tau, x, \xi)\, d\xi\, d\tau. \qquad \ldots(13)$$

Thus $A(\tau, x, \xi)$ describes the infectivity at x due to one infective of 'age of illness' τ at ξ. The above definitions and assumptions lead to the following system of equations

$$\frac{\partial S}{\partial t}(t, x) = -S(t, x)\, B(t, x) \qquad \ldots(14)$$

$$i(t, 0, x) = \frac{\partial s}{\partial t}(t, x) \qquad \ldots(15)$$

$$i(t: \tau, x) = i(t - \tau, 0, x). \qquad \ldots(16)$$

Using (15) and (16) to eliminate i from (13) and then substituting the result in (14) we have

$$\frac{\partial S}{\partial t}(t, x) = S(t, x) \int_0^\infty \int_\Omega \frac{\partial S}{\partial t}(t - \tau, \xi)\, A(\tau, x, \xi)\, d\xi\, d\tau. \qquad \ldots(17)$$

An appropriate initial value problem is obtained by prescribing

$$i(0, \tau, x) = i_0(\tau, x) \text{ and } S(0, x) = S_0(x)$$

and by restricting the validity of equations to $t > 0$. Then instead of (17), we obtain

$$\frac{\partial S}{dt}(t, x) = S(t, x) \left\{ \int_0^t \int_\Omega \frac{\partial S}{\partial t}(t - \tau, \xi) A(\tau, x, \xi) \, d\xi \, dt \right.$$

$$\left. - h(t, x) \right\} \qquad \ldots(18)$$

where

$$h(t, x) = \int_0^\infty \int_\Omega i_0(\tau, \xi) A(t + \tau, x, \xi) \, d\xi \, dt. \qquad \ldots(19)$$

Further, assuming $S_0(x) > 0$ for every $x \in \Omega$, integrate equation (18) with respect to t. By interchanging the order of the integrations, the resulting equation can be written in the form

$$u(t, x) = \int_0^t \int_\Omega g(u(t - \tau, \xi)) S_0(\xi) A(\tau, x, \xi) \, d\xi \, d\tau + f(t, x) \qquad \ldots(20)$$

where

$$u(t, x) = -\ln \frac{S(t, x)}{S_0(x)}, \; g(y) = 1 - e^{-y}, \; f(t, x) = \int_0^t h(\tau, x) \, d\tau.$$

We note that eqn. (20) is a special form of our eqn. (2). Hence, Theorem 1 can be applied to guarantee the existence and uniqueness of the solutions of eqn. (20) with suitable modifications in assumptions (H_1) and (H_2). It is easy to observe that, here we asserts the existence and uniqueness of the solutions of eqn. (20) under different conditions from those given by Diekmann[3].

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